Exam 3: If you need to take the conflict, or have accommodations, please email me ASAP.

Last time: Rate of flow of a vector field \( \vec{F} \) across a path \( C \) is given by \( \int_{C} \vec{F} \cdot \vec{n} \, ds \), \( \vec{n} \) a normal vector to \( C \).

\[ \int_{C} \vec{F} \cdot \vec{n} \, ds \] is called the flux of \( \vec{F} \) across \( C \).

How to find \( \vec{n} \): \( \vec{r}(u) \) parameterizes \( C \).

\[ \vec{n}(u) \] should be \( \perp \) to \( \vec{r}'(u) = (x'(u), y'(u)) \).

\[ \vec{n}(u) = \frac{(y'(u), -x'(u))}{|\vec{r}'(u)|}. \]
Compute the flux of \( \mathbf{F}(x,y) = (x,y) \) across the circle \((x-1)^2 + y^2 = 1\).

\[
\mathbf{r}(u) = (1 + \cos u, \sin u)
\]

\[
\mathbf{r}'(u) = (-\sin u, \cos u)
\]

Note: this already has length 1.

It points outwards from circle.

\[
\mathbf{n}(u) = (\cos u, \sin u)
\]

\[
\text{Flux} = \int_{c} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{0}^{2\pi} (1 + \cos u, \sin u) \cdot (\cos u, \sin u) \frac{1}{|\mathbf{r}'(u)|} \, du
\]

\[
= \int_{0}^{2\pi} (\cos u + \cos^2 u + \sin^2 u) \, du
\]

\[
= 2\pi.
\]

This tells us that on average, the net effect of the vector field flow is that fluid is leaving the circle.
Suppose $C$ is a closed curve in $\mathbb{R}^2$.
and $\vec{F}(x,y) = (P(x,y), Q(x,y))$.

$$\text{Flux} = \oint_C \vec{F} \cdot d\vec{s} = \int_C (P, Q) \cdot \left( \frac{y'(u)}{\sqrt{1 + (y'(u))^2}}, -\frac{x'(u)}{\sqrt{1 + (y'(u))^2}} \right) \, du.$$

$$= \int_C (-Q(x,y) \cdot x'(u) + P(x,y) y'(u)) \, du.$$

$$= \oint_C (-Q, P) \cdot d\vec{r}.$$

Green's Theorem: $\iint_D \left( \frac{\partial P}{\partial x} - (- \frac{\partial Q}{\partial y}) \right) \, dA$

$$= \iint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dA.$$

\text{Notation: } \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)

Then $\text{div} \vec{F} = \nabla \cdot \vec{F} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \cdot (P(x,y), Q(x,y))$

$$= \frac{\partial}{\partial x} P + \frac{\partial}{\partial y} Q$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \text{div} \vec{F}.$$
Divergence Theorem: For \( C \) a closed curve in \( \mathbb{R}^2 \) bounding a region \( D \), \( \mathbf{n} \) an outward-pointing normal, then flux of \( \mathbf{F} \) through \( C \) is:

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \text{div} \, \mathbf{F} \, dA.
\]

assumes \( C \) is fixed
measures net flow of \( \mathbf{F} \) into or out of \( C \).

assumes \( C \) is more like a balloon,
this measures rate of expansion of area or volume.