Last Time: If \( \overrightarrow{F}(x,y) \) is defined on an open, simply connected domain \( D \), and then

\[ \overrightarrow{F} \] is conservative if and only if \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \).

- If \( \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \) but \( \overrightarrow{F} \) is not conservative, then restricting \( \overrightarrow{F} \) to a simply connected domain makes it conservative.

  For \( \overrightarrow{F}(x,y) = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \), defined on \( \mathbb{R}^2 \), \( 1x > 0 \).

- If \( \overrightarrow{F}(x,y) \) has \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \) but \( \overrightarrow{F} \) is not defined on a simply-connected domain.

  Can we tell whether \( \overrightarrow{F} \) is conservative?

  No: Because we can take a conservative vector field defined on \( \mathbb{R}^2 \) and restrict to a non-simply-connected domain.

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**Ch. 15 Double Integrals.**

We did partial derivatives & find slopes.

**Calculate this volume.**

\[ R = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y) \, dy \, dx \]
Recall:

Area Estimate = $\sum_{i=1}^{N} f(x_i^*) \Delta x$

As $\Delta x \to 0$, the estimate $\to \int_{a}^{b} f(x) \, dx$.

[assuming $f$ is continuous]

Extend the same idea.

Volume Estimate

$= \sum_{j=1}^{m} \sum_{i=1}^{N} f(x_i^*, y_j^*) \Delta x \Delta y$

as $\Delta x, \Delta y \to 0$

$\Rightarrow \iiint f(x, y) \, dA$

$dA = dx \, dy = \text{area element}$. 
Alternative viewpoint:

\[ \text{Area} = \int_c^d f(x_0, y) \, dy. \]

Hold \( x \) constant, integrate w.r.t. \( y \) to get the area of a slice. \( \text{Area} = A(x) \) is a function of \( x \).

\[ \int_a^b A(x) \, dx \] is then the volume I want.

We can also do \( \int_a^b f(x_1 y_0) \, dx \) for area of slice. Then \( \int_c^d A(y) \, dy \) to find volume.

Volume = \( \int_a^b \left( \int_c^d f(x_1 y) \, dy \right) \, dx = \int_c^d \left( \int_a^b f(x_1 y) \, dx \right) \, dy \).

where integrating with respect to one variable means holding the other constant.

\( f(x_1 y) = 3. \quad R = \{ (x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2 \}. \)

\[ \text{Volume} = \int_a^b \int_c^d \]
Expect volume = length × width × height = 1 × 1 × 3 = 3.

Volume = \( \int_a^b \left( \int_c^d 3 \, dy \right) \, dx = \int_0^1 \left( \int_1^2 3 \, dy \right) \, dx \)

= \( \int_0^1 [3y]^2 \, dx = \int_0^1 3 \, dx = \left[ 3x \right]^1_0 = 3 \).

eg: \( f(x,y) = x^2 + y^2 + 1 \) \( R = \{(x,y) | 0 ≤ x ≤ 1, 0 ≤ y ≤ 1\} \)

Volume = \( \int_0^1 \left( \int_0^2 (x^2 + y^2 + 1) \, dy \right) \, dx = \int_0^1 \left[ xy^2 + \frac{1}{3}y^3 + y \right]_0^2 \, dx \)

= \( \int_0^1 \left[ (x^2 + \frac{1}{3} + 1) - (0) \right] \, dx = \int_0^1 \left( x^2 + \frac{4}{3} \right) \, dx \)

= \( \left[ \frac{1}{3}x^3 + \frac{4}{3}x \right]_0^1 = \frac{5}{3} \).
\( f(xy) = xe^{xy} \). \( R = \{(x,y) \mid 0 \leq x \leq 2, \ -1 \leq y \leq 3\} \)

or

\[
\int_{y=1}^{3} \left( \int_{x=0}^{2} xe^{xy} \, dx \right) \, dy
\]

\[
\int_{x=0}^{2} \left( \int_{y=1}^{3} xe^{xy} \, dy \right) \, dx = \int_{x=0}^{2} \left[ e^{xy} \right]_{y=1}^{3} \, dx
\]

\[
= \int_{x=0}^{2} (e^{3x} - e^{-x}) \, dx.
\]

\[
= \left[ \frac{1}{3} e^{3x} + e^{-x} \right]_{x=0}^{2}
\]

\[
= \frac{1}{3} e^{6} + e^{-2} - \left( \frac{1}{3} + 1 \right) = \frac{1}{3} e^{6} + e^{-2} - \frac{4}{3}
\]