Last time: Fundamental Theorem of Line Integrals:

If \( \mathbf{F} \) is conservative, then \( \int_c \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \)

i.e. \( \int_c \mathbf{F} \cdot d\mathbf{r} \) is independent of path (depends only on endpoints.)

\[ \text{ie a particle moving from } \mathbf{r}(a) \text{ to } \mathbf{r}(b) \text{ experiences the same net force regardless of path taken.} \]

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Q: How to tell when \( \mathbf{F} \) is conservative?

\( \diamond \) If \( \int_c \mathbf{F} \cdot d\mathbf{r} \) is not path independent, then \( \mathbf{F} \) is not conservative.

\[ \text{eg } \mathbf{F}(x,y) = (-y, x) \]

\[ \int_{C_1} \mathbf{F} \cdot d\mathbf{r} > 0 \]

\[ \int_{C_2} \mathbf{F} \cdot d\mathbf{r} < 0 \]

Integral is not path independent, so \((-y, x)\) is not conservative.
Suppose \( C \) is a closed curve (joins up on itself).

If \( \vec{F} \) is conservative,

\[
\oint_{C} \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{-C_2} \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r} = 0.
\]

If \( \vec{F} \) is conservative, then \( \oint_{C} \vec{F} \cdot d\vec{r} = 0 \) for all closed curves.

If there is a closed curve \( C \) so that \( \oint_{C} \vec{F} \cdot d\vec{r} \neq 0 \), then \( \vec{F} \) is not conservative.

**Example:**

\[
\oint_{C} \vec{F} \cdot d\vec{r} > 0
\]

since not 0, \( \vec{F} \) is not conservative.

**Example:**

\( \vec{F}(x,y) = (x y^3, x y) \). Try unit circle.

\[
\vec{r}(t) = (\cos t, \sin t) \quad \vec{r}'(t) = (-\sin t, \cos t).
\]

\[
\oint_{C} \vec{F} \cdot d\vec{r} = \int_{0}^{2\pi} (\cos t \sin^3 t, \cos t \sin t) \cdot (-\sin t, \cos t) \, dt
= \int_{0}^{2\pi} (-\cos t \sin^4 t + \cos^2 t \sin t) \, dt = 0.
\]
This alone is not enough to conclude \( F \) is conservative.

\[
\begin{align*}
\vec{r}_1(t) &= (t,t) & 0 \leq t \leq 1 & \vec{r}_1'(t) = (1,1) \\
\vec{r}_2(t) &= (t,t^2) & 0 \leq t \leq 1 & \vec{r}_2'(t) = (1,2t) \\
F(x,y) &= (xy^3, x^2y)
\end{align*}
\]

\[
\begin{align*}
\int_{C_1} \vec{F} \cdot d\vec{r} &= \int_0^1 (t^4, t^2) \cdot (1,1) \, dt = \int_0^1 t^4 + t^2 \, dt = \frac{8}{15} \\
\int_{C_2} \vec{F} \cdot d\vec{r} &= \int_0^1 (t^7, t^3) \cdot (1,2t) \, dt = \int_0^1 t^7 + 2t^4 \, dt = \frac{21}{40} \neq \frac{8}{15}
\end{align*}
\]

Integral depends on path, so \( F \) is not conservative.

Suppose \( F(x,y) = (P(x,y), Q(x,y)) \), and suppose \( \vec{F} = \nabla f \).

\[
\begin{align*}
\frac{\partial f}{\partial x} &= P(x,y) & \frac{\partial f}{\partial y} &= Q(x,y) \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial P}{\partial y} & \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial Q}{\partial x} \\
\frac{\partial^2 f}{\partial x \partial x} &= \frac{\partial P}{\partial x} & \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial Q}{\partial x} \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial P}{\partial x} & \frac{\partial^2 f}{\partial y \partial y} &= \frac{\partial Q}{\partial y}
\end{align*}
\]

If \( F \) is conservative, then \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \).
So e.g. \((-y, x)\) is not conservative because
\[
\frac{\partial P}{\partial y} = -1 \neq \frac{\partial Q}{\partial x} = 1
\]

e.g. \((xy^3, xy)\) is not conservative because
\[
\frac{\partial P}{\partial y} = 3xy^2 \neq \frac{\partial Q}{\partial x} = y
\]

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Q: Is \(\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\) enough to be conservative? Not quite.

eg \(\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)\)

\[
\frac{\partial P}{\partial y} = \frac{-(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}
\]

\[
\frac{\partial Q}{\partial x} = \frac{(x^2+y^2) - (x)(2x)y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}
\]

Integrate \(\vec{F}\) around unit circle. \(x^2+y^2=1\). \(\vec{r}(t) = (\cos t, \sin t)\)

\[
\int_0^{2\pi} (-\sin t, \cos t).(-\sin t, \cos t) \, dt
\]

\[
= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} 1 \, dt = 2\pi \neq 0.
\]
The problem is that $F$ is not defined at $(0,0)$.

More generally, $F$ is not defined on a simply connected domain.

A domain $D \subseteq \mathbb{R}^2$ is:

- **open** if its complement is closed, i.e., if $D$ contains none of its boundary points.
  
  
  \[ \text{eg } D = \{(x,y) | x^2 + y^2 < 1 \} \]

- **connected** if there is always a path in $D$ between two points.

- **simply connected** if it is connected and has no holes.

\[ \text{eg } \mathbb{R}^2 \text{ is simply connected} \]

\[ \mathbb{R}^2 \setminus \{(0,0)\} \text{ is not simply connected} \]
Theorem: Suppose $F$ is defined on an open, simply connected domain. Then

$F$ is conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. 