Last time: vectors are directed line segments.

- Addition
- Scalar multiplication
- Unit vectors have length 1.

12.3 Dot product.

\[ \vec{u} = (u_1, u_2, u_3) \quad \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 \]

\[ \vec{v} = (v_1, v_2, v_3) \]

- Sometimes called the scalar product.
- \( \vec{u} \cdot \vec{v} = ||\vec{u}|| ||\vec{v}|| \).
- \( \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \).
- \( c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) \).

Important formula:

\[ \vec{u} \cdot \vec{v} = ||\vec{u}|| ||\vec{v}|| \cos \theta \]

\[ 0 \leq \theta \leq \pi \]

is the angle between \( \vec{u} \) and \( \vec{v} \).

Proof uses Law of Cosines. (see textbook).
Consequence: Assuming \( \vec{u}, \vec{v} \) are not \( \perp \), then
\[ \vec{u} \cdot \vec{v} = 0 \] if and only if \( \vec{u} \) and \( \vec{v} \) are orthogonal.

Projection of vectors onto other vectors.

\[ \frac{\vec{u} \cdot \vec{v}}{\| \vec{u} \|} = \text{comp}_\vec{u} \vec{v} \]

"the component of \( \vec{v} \) along \( \vec{u} \)"

This distance is
\[ \| \vec{v} \| \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\| \vec{u} \|} \]

If we want a vector vector
\[ \text{proj}_\vec{u} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\| \vec{u} \|^2} \vec{u} \]

"the projection of \( \vec{v} \) onto \( \vec{u} \)"
Work done = \left( \overrightarrow{F.D} \right) \overrightarrow{D} \overrightarrow{D} \overrightarrow{D} \overrightarrow{D}

= \left( \frac{\overrightarrow{F.D}}{D^2} \right) \overrightarrow{D}

= 1 \overrightarrow{F.D}.

12.5 Lines and Planes

Q. How much information determines a plane uniquely?

A plane has a vector orthogonal to it, this is called a normal vector and denoted \( \overrightarrow{n} \).

Lots of planes are parallel to one another and have the same \( \overrightarrow{n} \). Also need one point on the plane.

\( \overrightarrow{n} = (a, b, c) \quad P_0 = (x_0, y_0, z_0) \) is a point in the plane.

\( P = (x, y, z) \) some other point in the plane.

\( \overrightarrow{P_0P} \) is \( \perp \) to \( \overrightarrow{n} \).
ie $(x-x_0, y-y_0, z-z_0) \cdot (a, b, c) = 0$.  
$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$.  
$ax + by + cz \equiv (ax_0 + by_0 + cz_0) = 0$.  
*just a number.*

Notice: Coefficients of $x, y, z$ respectively are components of $\mathbf{n}$.  

![Diagram of a normal vector and a plane]

Can you tell me a normal vector to the plane $2x - 3y + 6z = 12$?

1  
Thanks!

*You could also have* 
$(4, -6, 12)$ or 
$(-2, 3, -6)$.  

eg. \( z = 1 \) and \( x - y + z = 0 \).

How do they intersect? Note: They are not parallel.

\[ \vec{n}_1 = (0, 0, 1) \]
\[ \vec{n}_2 = (1, -1, 1) \]

Angle between planes = angle between \( \vec{n}_1 \) and \( \vec{n}_2 \).

\[ \cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{||\vec{n}_1|| \, ||\vec{n}_2||} = \frac{1}{1 \cdot \sqrt{3}} = \frac{1}{\sqrt{3}}. \]

\[ \theta = \cos^{-1} \left( \frac{1}{\sqrt{3}} \right) \approx 54.6^\circ \]

**Line of intersection**: Start with two points.

\( z = 1 \) and \( x - y + z = 0 \).

\((-1, 0, 1) \text{ and } (0, 1, 1)\).\n
\[ \vec{v} = (0, 1, 1) - (-1, 0, 1) \]
\[ = (1, 1, 0) \]

**Line** = \((-1, 0, 1) + t(1, 1, 1) = (-1 + t, t, 1)\).