Math 241 Honors Homework 3

due Thursday, November 15th, in class

Note: For this homework assignment, full proofs and clear explanations are required. Each problem should be solved on its own sheet(s) of paper. These problems are intended to be more challenging than the regular homework problems, so do not worry if you cannot solve them immediately. Because of this, it is recommended that you give yourself plenty of time to think about these problems. You can use any results from the textbook and the notes, but you may not use any other sources.

1. We talked about the proof of the Arithmetic-Geometric Mean inequality, using Lagrange Multipliers, in class. It states that, for $n$ positive integers $x_1, \ldots, x_n$,
\[
\sqrt[n]{x_1x_2\ldots x_n} \leq \frac{x_1 + \ldots + x_n}{n}.
\]
Use this to prove the following.
(a) For any positive real numbers $a_1, \ldots, a_n$, it is true that
\[
n \left( \frac{1}{a_1} + \ldots + \frac{1}{a_n} \right)^{-1} \leq (a_1a_2\ldots a_n)^{\frac{1}{n}}.
\]
(b) Given three positive real numbers $\alpha, \beta, \gamma$ so that $\alpha + \beta + \gamma = 1$, let two functions be defined by $W(x, y, z) = \alpha x + \beta y + \gamma z$ and $G(x, y, z) = (xyz)^{\frac{1}{3}}$. Show that there is a constant $c$ so that for any positive $x, y, z$ it is true that
\[
G(x, y, z) \leq cW(x, y, z).
\]
2. Evaluate
\[
\int_{x=0}^{a} \int_{y=0}^{b} e^{\max\{b^2x^2, a^2y^2\}} \, dy \, dx
\]
where $a$ and $b$ are positive real numbers and max denotes the larger of the two numbers $b^2x^2$ and $a^2y^2$.
(Hint: Use symmetry, and begin with the easier case $a = b = 1$.)

3. Solve problems 5 and 6 on page 1053 of the textbook. The purpose of these two problems is to use the double integral
\[
\int_0^1 \int_0^1 \frac{1}{1 - xy} \, dx \, dy
\]
and a change of variables to prove the famous equation
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]
4. Solve problem 3 of the Volumes of Hyperspheres discovery project on page 1027 of the textbook, which asks for a formula for the hypervolume of a 4-dimensional sphere. It may be helpful to try the first two problems as a warm up. For bonus credit, you can also try problem 4, which asks for the volume of an $n$-dimensional sphere.