Math 241 Honors Homework 1

due Thursday, September 20th, in class

Note: For this homework assignment, full proofs and clear explanations are required. Each problem should be solved on its own sheet(s) of paper. These problems are intended to be more challenging than the regular homework problems, so do not worry if you cannot solve them immediately. Because of this, it is recommended that you give yourself plenty of time to think about these problems. You can use any results from the textbook and the notes, but you may not use any other sources.

1. Consider the matrix

\[ A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \]

and the corresponding linear transformation \( T_A : \mathbb{R}^3 \to \mathbb{R}^3 \).

(i) Show that the transformation \( T_A \) fixes a line in \( \mathbb{R}^3 \). Find the angle which this line makes with each of the three coordinate axes.

(ii) Show that the transformation \( T_A \) is a rotation around this line. By what angle does \( T_A \) rotate vectors orthogonal to the line?

2. It is not true in general that given three vectors \( \vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3 \), we have

\[ \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}. \]

However, for some choices of vectors, this identity does hold (for example, if \( \vec{0} \) is one of the vectors). Produce a characterization of all triples of vectors that do satisfy the identity, and explain what is true geometrically about these vectors.

3. It is not true in general that given two 2 x 2 matrices \( A \) and \( B \), we have \( AB = BA \). However, it is true for some choices of matrices (for example, if one of the matrices has all entries equal to 0, then the identity holds).

(a) Prove that if \( A \) and \( B \) are both rotations about \( \vec{0} \), then the identity holds.

(b) There are, in fact, some choices for \( A \) so that no matter what the entries of \( B \) are, the identity still holds. Characterize these matrices, and explain what the corresponding linear transformations do when applied to vectors in \( \mathbb{R}^2 \).

4. Recall that the Cauchy-Schwarz inequality states that for two vectors \( \vec{a}, \vec{b} \in \mathbb{R}^n \), we have

\[ |\vec{a} \cdot \vec{b}| \leq ||\vec{a}|| ||\vec{b}||. \]

This is really an inequality regarding real numbers. Prove the following two inequalities by applying the Cauchy-Schwarz inequality to the appropriate choice of vectors.

(a) If \( a_1, \ldots, a_n \) are real numbers satisfying \( \sum_{i=1}^{n} a_i^2 = n \), then

\[ |a_1 a_2 + a_2 a_3 + \ldots + a_n a_1| \leq n. \]

(b) If \( a_1, \ldots, a_n \) are real numbers satisfying \( \sum_{i=1}^{n} a_i = 1 \), then

\[ \sum_{i=1}^{n} a_i^4 \geq \frac{1}{n^4}. \]