Some Math 221 Midterm 2 Review problems from old midterms

1. Find the derivative $dy/dx$ if $y = x^{\sin(3x)}$.

2. Find the derivative $dy/dx$ if $y^x = x^y$.

3. Find $dy/dx$ if $e^{5y} \tan x = 1 + \cos(xy^2)$.

4. Compute that the derivative of $f(x) = \arctan x$ is $f'(x) = \frac{1}{1+x^2}$ using implicit differentiation.

5. The equation $x^2 - xy + y^2 = 3$ represents a “rotated ellipse”. Find the points at which this ellipse crosses the x-axis and show that the tangent lines at these points are parallel.
6. Find the derivative of \( f(x) = e^{5x^2} \arctan(5x^2) \). No partial credit.

7. Find the derivative of \( f(x) = \log_3(3^x + 1) \). No partial credit.

8. Find the 1000th derivative of \( f(x) = xe^{-x} \).

9. If \( f'(x) = -5f(x) \) and \( f(0) = 10 \), find \( f(x) \).

10. A bacteria population starts at time \( t = 0 \) with 700 cells and doubles every three hours. How fast is the population GROWING at time 6 hours? Leave your answer as an exact value; do not approximate.
11. Use linear approximation or differentials to find $\sqrt{103}$. Is this an overestimate or an underestimate? Use concavity to explain.

12. Use Newton’s method for one step to approximate $\sqrt[3]{1004}$.

13. If $f(x) = 3\sqrt{1 + x^3}$, find the differential $df$ if $x = 2$ and $dx = 1/10$.

14. Approximately how much paint would you need to cover a sphere of radius 10 centimeters with a 0.1 millimeter coat of paint? A sphere with radius $r$ has volume $V = \frac{4}{3}\pi r^3$.

15. Suppose that the line $y = 5x - 4$ is tangent to the curve $y = f(x)$ when $x = 3$. If Newton’s method is used to locate a root of the equation $f(x) = 0$, and the initial approximation is $x_1 = 3$, find the second approximation.
16. State the Mean Value Theorem precisely and illustrate with a sketch.

17. State the Intermediate Value Theorem precisely and illustrate with a sketch.

18. State Rolle’s theorem precisely and illustrate with a sketch.

19. Using theorems from class, prove that $x^3 + 5x + 1 = 0$ has exactly one root.

20. Suppose you are driving due north with continuous position and velocity functions. At $t = 0$, you start from a dead stop. After two hours, you have driven 60 miles. Must your velocity ever have been exactly 30 mph? How about 15 mph? You must use theorems to explain; no credit for intuitive answers.
21. Use the first derivative test to find all local maxima and minima of \( f(x) = x^{1/3}(1-x) \). Hint: multiply it out first.

22. Find all local maxima, minima, and inflection points of \( f(x) = 4x^5 - 5x^4 \) and give a rough sketch of the graph.

23. Find all local maxima, minima, and asymptotes of \( A(x) = \frac{100+x^2}{x} \) and give a rough sketch of the graph. Hint: simplify \( A \) first.

24. Graph \( f(x) = x^2 \ln x \), indicating all local maxima and minima, intercepts, and limiting behaviour.
25. Here is a sketch of \( f'(x) \), THE DERIVATIVE OF \( f(x) \). (For the review sheet, draw your favorite \( f' \) here.) Note: this is NOT a sketch of \( f(x) \).

(a) On which intervals is \( f(x) \) increasing? (ie, where is \( f'(x) \geq 0 \)?)

(b) Sketch \( f(x) \) accurately, indicating any local minima or maxima and inflection points.

26. In despair over your WebAssign homework, you throw your computer off a 144 foot cliff. If its position is given by \( h(t) = 144 - 16t^2 \), where \( h \) is in feet and \( t \) is in seconds, find its velocity and acceleration just before it hits the ground. Please include units in your answers.

27. Find the mistake(s), if any, in the following calculation:

\[
\lim_{x \to 0} \frac{\cos x}{x^2} = \lim_{x \to 0} \frac{-\sin x}{2x} = \lim_{x \to 0} \frac{-\cos x}{2} = \frac{1}{2}.
\]

28. Find the following limits.

(a) \( \lim_{x \to 0} \left( \frac{x^3}{\sin x - x} \right) \)

(b) \( \lim_{x \to \infty} \ln(5 + (x^{10}/e^x)) \)

(c) \( \lim_{x \to 1} \left( \frac{3}{\ln(x)} - \frac{3}{x-1} \right) \)
29. The height of a cylinder is decreasing at a rate of 3cm/min while its radius is increasing at a rate of 2cm/min. At what rate is the volume of the cylinder changing when the height is 20cm and the radius is 10cm?

30. Two men start at the same point. One walks north at 3 miles per hour and the other walks east at 4 miles per hour. How fast is the distance between them growing when the first man is 3 miles north and the second is 4 miles west?

31. Suppose an ostrich 8 ft tall is walking at a speed of 2 ft/s directly toward a street light 12 ft high. How fast is the tip of his shadow moving along the ground when he is 6 feet from the base of the lampost?
32. A cylindrical can with both a top and a bottom needs to contain $V$ cubic centimeters. Find the dimensions that will minimize the cost of the metal to make the can, assuming that the sides and top and bottom are made out of the same thickness of material.

33. A 10x20 piece of cardboard is to be turned into a box by removing squares from each corner and folding the resulting flaps up. Find the dimensions that maximize the amount of volume contained in the resulting box.
34. If 121 square centimeters of material is available to make a box with a **square base** and an **open top**, find the largest possible volume of the box. **Hint:** the box has four sides and one bottom.

35. Find approximate values for the coordinates of the point on the hyperbola $xy = 1$ closest to the point $(3, 0)$. **Hint:** use Newton’s method one step (starting from a good initial choice) to approximate the critical point.