Due Wednesday, November 14, 2018.

1. Given a network $G$ with capacities $c$, and a feasible flow $f$ on $G$, suppose there is no path in the residual graph $G_f$ from $s$ to $t$. Let $S$ be the set of vertices reachable from $s$ along a directed path in $G_f$, and let $T = V \setminus S$.

(a) Show that $f$ saturates every edge from $S$ to $T$ and avoids every edge from $T$ to $S$.

(b) Explain why in this case $f$ must be a maximal flow for $G$ and $(S, T)$ must be a minimum cut for $G$.

2. Use the Ford Fulkerson method, starting from the zero flow, to find the maximum flow and minimum cut for the network with capacities shown below.

After each augmenting step, list the augmenting path as a sequence of vertices, and redraw the graph $G$ with the new flow/capacity along each edge. For instance, the graph with zero flow is written below. (If you are TeXing up your homework, send me email and I’ll send you the tikzpicture template I’m using).

3. Suppose there are $n$ students, $n$ internships, and a team of $m$ recruiters. Each recruiter has a list of students and internships as clients, and they can assign any student on their client list to any internship on their client list. A student can be on more than one recruiter’s client list, and an internship can be on more than one recruiter’s client list, but each student can be assigned to at most one internship and each internship can be assigned to at most one student. The students and internships agree to take whoever or whatever the team of recruiters supplies, without question. In addition, we restrict the number of assignments that recruiter $j$ can arrange to a maximum of $b_j$.

Translate the problem of finding the most possible student-internship assignments into one of finding the maximum flow in a flow network. You must specify the vertices, edges, and edge capacities of the network.

4. Suppose we are given a 3x3 matrix $F$ of non-negative real numbers between 0 and 1. We want to round $F$ to an integer matrix by replacing each entry $x$ in $F$ with either $\lfloor x \rfloor$ or $\lceil x \rceil$, without changing the sum of entries in any row or column of $F$. For example:

$$
\begin{bmatrix}
.2 & .4 & .4 \\
.9 & .0 & .1 \\
.9 & .6 & .5
\end{bmatrix} \mapsto
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1
\end{bmatrix}
$$

Use network flows to write an algorithm that either rounds $F$ in this fashion or reports that no such rounding exists. Answer all three questions (on the next page).
(a) Specify the vertices, edges, and edge capacities of the network.

(b) Show that an integral flow that saturates every edge leaving the source gives a legal rounding of the matrix.

(c) Show that every legal rounding corresponds to a flow that saturates every edge leaving the source.