Due Wednesday, October 17, 2018

1. Given the LP below

Maximize \( z = 2x_1 + x_2 \)

subject to

\[
\begin{align*}
  x_1 + x_2 + x_3 &= 3 \\
  x_1 - x_2 + x_4 &= 2 \\
  x_1, x_2, x_3, x_4 &\geq 0.
\end{align*}
\]

Which of the following bases are primal feasible? Which are dual feasible? Are any of them optimal?
(a) \( B = (1, 4) \)
(b) \( B = (2, 3) \)
(c) \( B = (1, 2) \)

2. Use the dual simplex method to find an optimal solution to the problem

Minimize \( z = 7x_1 + x_2 + 3x_3 + x_4 \)

subject to

\[
\begin{align*}
  2x_1 - 3x_2 - x_3 + 2x_4 &\geq 8, \\
  -x_1 + x_2 + x_3 - x_4 &\geq 10, \\
  x_1, x_2, x_3, x_4 &\geq 0.
\end{align*}
\]

Next, suppose the objective function in is changed to

Minimize \( z = x_2 + 3x_3 + x_4 \)

Solve the new problem starting from the optimal tableau from the first part of the question.

3. Suppose that you have the LP in standard form

\[
(P) \quad \min \quad c^T x \quad \text{s.t.} \quad Ax = b, \quad \text{and} \quad x \geq 0
\]

and a dual feasible basis \( B = (j_1, \ldots, j_m) \) such that

\[
A_B^{-1}b = \begin{pmatrix} 18 & 12 & 8 & 9 & 31 \end{pmatrix}^T, \quad y^T = c_B^T A_B^{-1} = \begin{pmatrix} 8 & 20 & 30 & 7 & 8 \end{pmatrix}, \quad \text{and} \quad y^T b = 159.
\]

Suppose that \( e_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T \) is the third standard basis vector and that the third column of \( A_B^{-1} \) is \( \begin{pmatrix} 1 & 5 & -2 & 2 \end{pmatrix}^T \).
Find the largest number $\Delta$ such that if $b' = b + \Delta e_3$, then $B$ is both a primal and dual feasible basis for this new LP:

$$(P') \quad \min c^T x \text{ s.t. } Ax = b'.$$

Also, with the maximum value of $\Delta$ that was computed in the previous step, find the value of the objective function at an optimal solution of $P'$.

4. Let (P) be the following LP

$$\min c^T x \text{ s.t. } Ax \geq -c \text{ and } x \geq 0 \quad (P)$$

where $A$ is skew-symmetric, which means that $A = -A^T$. Prove that if there exists a feasible solution of $P$, then there exists an optimal solution $x^*$ of $P$ and $c^T x^* = 0$. 