Theorem 1. If a set \(\{v_1, \ldots, v_n\}\) spans the vector space \(V\), then any set \(\{w_1, \ldots, w_p\}\) of nonzero vectors in \(V\) with more elements (i.e., \(p > n\)) must be linearly dependent.

Proof. Suppose that \(\{w_1, \ldots, w_p\}\) are linearly independent and \(\{v_1, \ldots, v_n\}\) span \(V\). Then \(w_1\) can be written as
\[
  w_1 = a_1v_1 + a_2v_2 + \cdots + a_nv_n
\]
for some scalars \(a_1, \ldots, a_n\), where at least one of the \(a_i\) is nonzero since \(w_1 \neq 0\). By reordering the \(v\)'s if necessary, we have that \(a_1 \neq 0\), and
\[
  v_1 = \frac{1}{a_1} w_1 - \frac{a_2}{a_1} v_2 - \cdots - \frac{a_n}{a_1} v_n.
\]
Since the set \(\{v_1, \ldots, v_n, w_1\}\) spans \(V\) and \(v_1\) is a linear combination of the other vectors in the set, the Spanning Set Theorem says that \(\{v_2, \ldots, v_n, w_1\}\) also spans \(V\).

Next, since \(\{v_2, \ldots, v_n, w_1\}\) spans \(V\),
\[
  w_2 = b_2v_2 + b_3v_3 + \cdots + b_nv_n + b_1w_1
\]
for some scalars \(b_1, \ldots, b_n\). Since the set \(\{w_1, w_2\}\) is linearly independent, at least one of \(b_2, \ldots, b_n\) must be nonzero. By reordering the \(v\)'s again if necessary, we have that \(b_2 \neq 0\), and
\[
  v_2 = \frac{b_1}{b_2} w_1 - \frac{b_3}{b_2} v_3 - \cdots - \frac{b_n}{b_2} v_n
\]
and so the set \(\{v_3, \ldots, v_n, w_1, w_2\}\) spans \(V\).

Continuing in this way we eventually find that the set \(\{w_1, \ldots, w_n\}\) spans all of \(V\), so \(w_{n+1}\) can be written as a linear combination of \(w_1, \ldots, w_n\). \(\square\)

Corollary 1. If a vector space \(V\) has a basis of size \(n\), then any other basis of \(V\) must also have size \(n\).

Proof. Suppose the sets \(\{v_1, \ldots, v_n\}\) and \(\{w_1, \ldots, w_p\}\) are both bases of \(V\), so both sets span and both sets are linearly independent. By the theorem above, \(p \leq n\), and symmetrically, \(n \leq p\). \(\square\)