Riemann Sums and Definite Integrals

Note: Don’t forget the local time change on Sunday: the minute after 1:59am will be 1:00am.

1. Evaluate the sum:
\[ \sum_{i=1}^{4} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \]

4. Write the sum in sigma notation:
\[ \frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27} = \sum_{i=3}^{25} \frac{1}{i+4} \]

2. Evaluate the sum.
\[ \sum_{j=2}^{n} (-1)^{j} = (-1)^{1} + (-1)^{2} + \cdots + (-1)^{n} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases} \]

5. Write the sum in sigma notation:
\[ 1 + 2 + 4 + 8 + 16 + 32 = \sum_{i=0}^{5} 2^i \]

3. Evaluate the sum.
\[ \sum_{i=0}^{3} 2^i = 2^0 + 2^1 + 2^2 + 2^3 = 15 \]

6. Write the sum in sigma notation:
\[ \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7} = \sum_{i=3}^{7} \sqrt{i} \]

7. For \( f(x) = e^{-x} \) on the interval \([0, 6]\), write the Riemann sum using 3 rectangles and left endpoints \((L_3)\), and draw a picture of the graph of the function and of the rectangles. Is \( L_3 \) an overestimate or an underestimate of the area under the graph?

\[ \Delta x = \frac{6-0}{3} = 2 \]

\[ L_3 = 2 \cdot f(0) + 2 \cdot f(2) + 2 \cdot f(4) \]
\[ = 2 \cdot e^0 + 2 \cdot e^{-2} + 2 \cdot e^{-4} \]
\[ = 2(1 + e^{-2} + e^{-4}) \]
\[ \text{overestimate} \]

8. For \( f(x) = \ln x \) on the interval \([0.5, 4.5]\), find the Riemann sum using 4 rectangles and midpoints \((M_4)\) and draw a picture of the graph of the function and of the rectangles.

\[ \Delta x = \frac{4.5-0.5}{4} = 1 \]

\[ M_4 = 1 \cdot f(1) + 1 \cdot f(2) + 1 \cdot f(3) + 1 \cdot f(4) \]
\[ = 1 \cdot \ln(1) + 1 \cdot \ln(2) + 1 \cdot \ln(3) + 1 \cdot \ln(4) \]
\[ = \ln(2) + \ln(3) + \ln(4) \]
\[ = \ln(24) \]
(9) A car is traveling at 60 feet per second when the driver sees a deer in the road 300 feet ahead and immediately steps on the brakes. The deer freezes and does not move from his spot in the road. I've recorded the driver's speed (in ft/sec) every two seconds starting at the time that he first stepped on the brakes and going until the time that the car finally came to a stop. Does the car hit the deer? Explain your reasoning.

<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>car's speed</td>
<td>60</td>
<td>46</td>
<td>28</td>
<td>12</td>
<td>0</td>
</tr>
</tbody>
</table>

Want an upper bound on distance traveled.

Since the speed is decreasing, approximation using left endpoints overestimates the distance.

\[
L_1 = 2 \cdot (60) + 2 \cdot (46) + 2 \cdot (28) + 2 \cdot (12)
\]

= 292 ft

Therefore, distance traveled \( \leq 292 \) ft.

Because 292 < 300, the car does not hit the deer.

Compute the following integrals using geometry.

(10) \( \int_2^8 7 \, dx \)

\( \int_2^8 7 \, dx \)

(11) \( \int_2^8 2x \, dx \)

\( \int_2^8 2x \, dx \)

(12) \( \int_2^8 7 + 2x \, dx \)

\( \int_2^8 7 + 2x \, dx \)

(13) \( \int_2^8 7 - 2x \, dx \)

\( \int_2^8 7 - 2x \, dx \)

(14) Compute the definite integral \( \int_2^8 2x \, dx \) by computing the limit of the Riemann sums as the number of rectangles goes to \( \infty \).

\[
\Delta x = \frac{8 - 2}{n} = \frac{6}{n}, \quad f(x) = 2x, \quad a = 2
\]

\[
\int_2^8 2x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(2 + (i - 1) \cdot \frac{6}{n}) \cdot \frac{6}{n}
\]

\[
= \lim_{n \to \infty} \frac{6}{n} \left[ (2 + \frac{6}{n}) \cdot \frac{6}{n} + (2 + \frac{12}{n}) \cdot \frac{6}{n} + \ldots + (2 + \frac{6n}{n}) \cdot \frac{6}{n} \right]
\]

= \( \lim_{n \to \infty} \left[ \frac{12}{n} \cdot (2n) + \frac{12}{n} \cdot (\frac{3n}{2}) \right] \)

= \( \lim_{n \to \infty} \left[ 24 + \frac{72}{n} \cdot \left( \frac{n(n+1)}{2} \right) \right] \)

= \( \lim_{n \to \infty} \left[ 24 + \frac{36}{n} \cdot (n^2+n) \right] \)

= \( \lim_{n \to \infty} (24 + 36 + \frac{36}{n}) \)

= \( 60 \)
(15) True or False, and Explain. Assume all the functions inside the integrals are integrable.

(a) \( \int_1^2 f(x)dx - \int_3^2 f(x)dx = \int_1^3 f(x)dx \)

(b) \( \int_0^5 f(x)dx = \int_0^5 f(\oplus)\oplus \)

(c) If \( m \leq f(x) \leq M \) for all \( x \) in \([a, b]\), then \( m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \).

(d) \( \int_0^2 (f(x) + 5g(x)) \, dx = \int_0^2 f(x)dx + 5 \int_0^2 g(x)dx \)

(e) \( F(x) = \int_0^x f(t)dt \) is a function of \( x \), not \( t \).