Newton's Method

1. Deriving the formula for Newton's Method.

(a) If $f$ is a differentiable function, what is the equation of the tangent line to the graph of $f$ at the point $x = x_0$?

\[
y - f(x_0) = f'(x_0)(x - x_0)
\]

\[
y = f'(x_0) \cdot x + f(x_0) - x_0 f'(x_0)
\]

(b) Determine the $x$-intercept of the tangent line to the graph of $f$ at the point $x = x_0$.

Let $y = 0$, we have

\[
f'(x_0) x = x_0 f'(x_0) - f(x_0)
\]

\[
x = x_0 - \frac{f(x_0)}{f'(x_0)}
\]

(c) Which one of the following correctly shows how $x_n$ can be used to determine the next approximation $x_{n+1}$ using Newton's Method?

- (a) $x_{n+1} = \frac{x_n + f(x_n)}{f'(x_n)}$
- (b) $x_{n+1} = x_n + \frac{f'(x_n)}{f(x_n)}$
- (c) $x_{n+1} = \frac{x_n + f(x_n)}{f'(x_n)}$
- (d) $x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$
- (e) $x_{n+1} = \frac{x_n - f(x_n)}{f'(x_n)}$
- (f) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- (g) $x_{n+1} = \frac{x_n - f(x_n)}{f'(x_n)}$
- (h) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

2. For which initial values in $[-1, 3]$ do you suspect

Newton's method will be successful in approximating

the root of the function graphed? Explain why

the method is likely to fail at the other values.

\[
\text{Succeed}
\]

$x_1 < 0$

(or $x_1 < 0.1$)

\[
\text{Fail}
\]

when $x_1$ is close to 0.2,

$f'(x_1)$ is close to 0,

(slope of tangent is 0 or close to 0)
3. Here is a 2000 year old iterative algorithm for computing the square root of a number \(c > 0\):

\[
x_{n+1} = \frac{1}{2}(x_n + c/x_n)
\]

The intuition is that if your guess \(x_n\) is too small, then \(c/x_n\) will be too big (and vice versa), so the average of \(x_n\) and \(c/x_n\) is likely to be a better approximation than \(x_n\) itself.

Show that this algorithm is just a restatement of Newton’s method applied to \(x^2 - c = 0\).

By Newton’s method, let \(f(x) = x^2 - c\), then \(f'(x) = 2x\), we have

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - c}{2x_n}
\]

\[
= x_n - \frac{1}{2}x_n + \frac{1}{2} \frac{c}{x_n} = \frac{1}{2} \left(x_n + \frac{c}{x_n}\right)
\]

4. Use Newton’s method to estimate \(\sqrt{5}\) to at least 4 decimal places.

Let \(f(x) = x^2 - 5\), then \(f'(x) = 2x\),

pick \(x_0 = 2\), then \(x_1 = 2 - \frac{2^2 - 5}{4} = \frac{9}{4}\)

\[
x_2 = \frac{9}{4} - \frac{\left(\frac{9}{4}\right)^2 - 5}{2 \times \frac{9}{4}} = \frac{161}{72} = 2.23611
\]

\[
x_3 = \frac{161}{72} - \frac{\left(\frac{161}{72}\right)^2 - 5}{2 \times \left(\frac{161}{72}\right)} = \frac{51841}{23184} = 2.23607
\]

Thus \(\sqrt{5} \approx 2.2361\)

5. (4.8 number 39) Use Newton’s method to find the x-coordinate, correct to four decimal places, of the point on the parabola \(y = (x - 1)^2\) that is closest to the origin.

Let \(D\) be the distance between \((0, 0)\) and \((x, (x-1)^2)\),

then \(D^2 = x^2 + ((x-1)^2 - 0)^2 = x^2 + (x-1)^4\). Let \(g(x) = x^2 + (x-1)^4\).

Then \(g'(x) = 2x + 4(x-1)^3\). We aim to find critical points for \(g(x)\), i.e. \(g'(x) = 0\).

Since \(g''(x) = 2 + 12(x-1)^2\). By Newton’s method, we have

pick \(x_0 = 0\), then \(x_1 = 0 - \frac{g'(0)}{g''(0)} = \frac{2}{7}\),

\[
x_2 = \frac{2}{7} - \frac{g'(\frac{2}{7})}{g''(\frac{2}{7})} = \frac{350}{1393} \approx 0.254983
\]

\[
x_3 = x_2 - \frac{g'(x_2)}{g''(x_2)} = 0.41008
\]

\[
x_4 = x_3 - \frac{g'(x_3)}{g''(x_3)} = 0.41025
\]