SECOND DERIVATIVE TEST AND CURVE SKETCHING

(1) Let \( f(x) = x^4 - 4x^3 + 4x^2 - 10 \). Find the critical points of \( f \) and determine if they are maxima or minima using the second derivative test.

\[
f'(x) = 4x^3 - 12x^2 + 8x = 4x(x-1)(x-2) \Rightarrow \text{Critical points at } x = 0, 1, 2
\]

\[
f''(x) = 12x^2 - 24x + 8 = 4(3x^2 - 6x + 2)
\]

\[
f''(0) > 0, \quad f''(1) < 0, \quad f''(2) > 0 \Rightarrow \text{local max at } x=1,
\]

\[
\text{local min at } x=0, 2.
\]

(2) Let \( f(x) = \frac{1}{x} - \frac{3}{x^3} \)

(a) Find the critical points of \( f \) and determine if they are maxima or minima using the second derivative test.

(b) Sketch the graph, labelling any roots, inflection points, and asymptotes.

(c) What kind of symmetry does the graph have, and how could you have predicted that symmetry?

\[
(a) \quad f' = \frac{-1}{x^2} + \frac{9}{x^4} = 0 \Rightarrow \text{Critical pts at } x = \pm 3
\]

\[
f'' = \frac{2}{x^3} - \frac{36}{x^5}, \quad f''(3) < 0, \quad f''(-3) > 0
\]

\[
(b) \quad f'' = 0 \Rightarrow x = \pm 3, \quad \text{inflection pts}
\]

\[
f'' < 0 \Rightarrow \text{roots at } x = \pm \sqrt{3}
\]

\[
\lim_{x \to 0^-} \frac{1}{x} - \frac{3}{x^3} = \lim_{x \to 0^+} \frac{x^2 - 3}{x^3} = \lim_{x \to 0^-} \frac{-3}{x^3} = -\infty, \quad \lim_{x \to 0^+} f(x) = -\infty
\]

(3) Show that the graph of \( y = x^{-1/3} \) has a sharp kink at \( x = 0 \) by finding \( \lim_{x \to 0^+} f'(x) \) and \( \lim_{x \to 0^-} f'(x) \).

\[
f' = \frac{2}{3} x^{-\frac{4}{3}}
\]

\[
\lim_{x \to 0^+} \frac{2}{3} x^{-\frac{4}{3}} = +\infty, \quad \lim_{x \to 0^-} \frac{2}{3} x^{-\frac{4}{3}} = -\infty
\]

(4) Suppose your patient has a fever \( T(t) \), and \( T' \) has been positive for a while. Would you be more worried if \( T''(0) < 0 \) or \( T''(0) > 0 \)? Sketch a possible graph of \( T \) for both cases.

\[
T'' < 0
\]

\[
T'' > 0 \Rightarrow \text{MORE WORRIED}
\]

\[
T' > 0
\]

\[
T > 0
\]
(5) TRUE or FALSE

(a) If $f$ has a local maximum at $c$, then $f'(c) = 0$.

\[ \text{F} \]

(b) If $f$ and $g$ are increasing on an interval $I$, then $f - g$ is increasing on $I$.

\[ \text{F} \]

(c) If $f$ and $g$ are positive increasing functions on an interval $I$, then $fg$ is increasing on $I$.

\[ \text{T} \]

(d) If $f$ has an inflection point which occurs at $x = c$, then $f''(c) = 0$.

\[ \text{T} \]

(e) If $f''(c) = 0$, then $f$ has an inflection point which occurs at $x = c$.

\[ \text{F} \]

(f) If $f$ is continuous and if $f'(x) < 0$ for all $x < 1$ and $f'(x) > 0$ for all $x > 1$, then $f$ has an absolute minimum at $x = 1$.

\[ \text{T} \]

(g) If $f'(7) = 0$ and if $f''(7) = 0$, then $f$ has neither a local minimum nor a local maximum at $x = 7$.

\[ \text{F} \]

(h) If $f'(5) = 0$ and if $f''(5) = -6$, then $f$ has a local maximum at $x = 5$.

\[ \text{T} \]