LINEAR APPROXIMATION AND DIFFERENTIALS

Recall the Very Useful Fact: If $f$ is differentiable at $a$, then $f(x) = f(a) + f'(a)(x - a)$ for $x$ near $a$.

We start by deriving some important linear approximations from scratch using the Very Useful Fact.

1. Compute the linear approximation of $\sin(x)$ at $x = 0$ using the Very Useful Fact.

$$f(x) = \sin x \quad f'(x) = \cos x \quad f(0) = 0 \quad \Rightarrow \quad f(x) \approx f(0) + f'(0)(x - 0)$$

$$\sin x \approx 0 + 1(x - 0)$$

2. Compute the linear approximation of $e^x$ at $x = 0$ using the Very Useful Fact.

$$f(x) = e^x \quad f'(x) = e^x \quad f(0) = 1 \quad \Rightarrow \quad f(x) \approx f(0) + f'(0)(x - 0)$$

$$e^x \approx 1 + 1(x - 0)$$

3. Compute the linear approximation of $(1 + x)^r$ at $x = 0$, where $r$ is any real number, using the Very Useful Fact.

$$f(x) = (1 + x)^r \quad f'(x) = r(1 + x)^{r-1} \quad f(0) = 1 \quad \Rightarrow \quad f(x) \approx f(0) + f'(0)(x - 0)$$

$$(1 + x)^r \approx 1 + r(x - 0)$$

4. Estimate $\sqrt[3]{8.1}$ by using your result from question 3. Is your estimate an underestimate or an overestimate?

$$\sqrt[3]{8.1} = (8 + \frac{1}{10})^{\frac{1}{3}} = 8^{\frac{1}{3}}(1 + \frac{1}{80})^{\frac{1}{3}} \approx 2(1 + \frac{1}{3} \cdot \frac{1}{80}) = 2.1$$

Since the tangent line lies above the graph, the approximation is an overestimate.

5. Estimate $e^{0.3}$. Is your estimate an underestimate or an overestimate?

$$e \approx 1 + 0.3 = 1.3$$

Since the tangent line is below the graph, the approximation is an underestimate.
Recall that if $f$ is differentiable at $a$, we define the differentials at $a$ to be the independent variable $dx$ and the dependent variable $df = f'(a)dx$, and we have that $f(a + dx) \approx f(a) + df$.

We also have the following rules for using differentials:
\[
\begin{align*}
d(u + v) &= du + dv \\
d(uv) &= udv + udv \\
d(u/v) &= (vdu - udv)/v^2 \\
d(u^n) &= ku^{n-1}du
\end{align*}
\]

6. (Section 3.10, Problem 35) The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm. Use differentials to estimate the maximum error if the surface area is calculated using this measurement.

\[
\begin{align*}
C &= 2\pi r \\
\frac{dC}{dr} &= 2\pi \\
\frac{dC}{dr} &= 2\pi \\
S &= 4\pi r^2 \\
\frac{dS}{dr} &= 8\pi r \\
\frac{dS}{dr} &= 8\pi r
\end{align*}
\]

\[dS = 4\pi (2.0) (0.5) \approx 26.8\]

7. A window has the shape of a square surmounted by a semi-circle. The base of the window is measured as having width 60 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum error possible in computing the area of the window.

\[dA = \left[1 + \frac{\pi}{8}\right] 2w dw\]

\[\text{Plug in } w = 60, \, dw = 0.1\]

\[dA = \left[1 + \frac{\pi}{8}\right] 2(60)(0.1)\]

\[= 16.07\]

8. Here are two ways to find the linear approximation for $\sqrt{9 + \sin x}$ for $x$ near 0.

(a) Calculate it directly using the Very Useful Fact.

(b) Combine two of the approximations from the start of the worksheet.

Do the two methods give the same approximation?

(a) \[f(x) = \sqrt{9 + \sin x}, \quad f(0) = 3, \quad f'(0) = \frac{1}{6}\]

\[\Rightarrow f(x) \approx 3 + \frac{1}{6}x\]

(b) \[\sqrt{9 + \sin x} \approx \sqrt{9 + x} = \sqrt{9 + \frac{x}{1 + \frac{1}{2}} \approx 3 + \frac{1}{6}\]}

\[\Rightarrow \text{[Same!]}\]