Limit Definition of Derivative

Instructions. Please work on these in groups, and then finish them on your own and upload them.

1. State the limit definition of the derivative, and sketch a picture to explain where this definition comes from.

\[ f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

2. The following are the graphs of the position of two racers.

Which racer has a faster average speed? When was each racer running faster than the other (approximately)? Which racer was running the fastest at the end of the race? Explain your answers.

3. The derivative \( f'(a) \) is which of the following? Pick all options that apply.

(a) An average rate of change
(b) An instantaneous rate of change
(c) A secant line
(d) A tangent line
(e) The slope of a secant line
(f) The slope of a tangent line

4. Let \( f(x) = x^3 \).

(a) Use the limit definition to find the derivative of \( f(x) \) at the point \( x = 1 \).

\[ \lim_{h \to 0} \frac{(1+h)^3 - 1^3}{h} = \lim_{h \to 0} \frac{1+3h+3h^2+h^3 - 1}{h} = \lim_{h \to 0} \frac{3h+3h^2+h^3}{h} = \lim_{h \to 0} (3+3h+h^2) = 3 \]

(b) Find the tangent line to the curve \( f(x) \) at the point \((1,1)\) and sketch the graph of the function and the tangent line.

\[ y - 1 = 3(x - 1) \Rightarrow y = 3x - 2 \]

(c) At \( x = 1 \), is \( f(x) \) increasing, decreasing, or neither? Explain.

\( f \) is increasing

(derivative is positive)
5. A dragonfly’s top velocity is nearly 1 kilometer per minute.

(a) Suppose the dragonfly’s velocity at time \( t = 0 \) is 0.6 kilometers per minute. How far does it travel in the first one tenth of a second?

\[
\begin{align*}
\text{Velocity at } t = 0 & = 0.6 \text{ km/min} = \left( \frac{6}{10} \right) \left( \frac{1}{60} \right) = \frac{1}{100} \text{ km/sec} \\
\text{Distance at } t = \frac{1}{10} & = \frac{1}{100} \left( \frac{1}{10} \right) = \frac{1}{1000} \text{ km} = 1 \text{ m}
\end{align*}
\]

(b) The dragonfly can’t keep up that pace for very long, however. Would you expect to be able to figure out how far the dragonfly went in thirty seconds from its velocity at \( t = 0 \)? Explain using a picture.

6. Let \( f(x) = \frac{2}{x} \)

(a) Use the limit definition to find the derivative of \( f(x) \) at the point \( x = 2 \).

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2 - \frac{2}{2+h}}{(2+h)h} = \lim_{h \to 0} \frac{-1}{2+h} = -\frac{1}{2}
\]

(b) Find the tangent line to the curve \( f(x) \) at the point (2,1) and sketch the graph of the function and the tangent line.

\[
\begin{align*}
y - 1 &= -\frac{1}{2} (x - 2) \\
y &= -\frac{1}{2} x + 2
\end{align*}
\]

(c) One important use of tangent lines is to approximate curves. For instance, the tangent line above is a good approximation to the curve \( y = \frac{2}{x} \), at least near the point (2,1). Use your tangent line to approximate the value of \( f(2.1) \).

\[
f(2.1) \approx f(2) + \left[ -\frac{1}{2} \right] (2.1 - 2) = 1 - \frac{1}{2} (0.1) = 1 - \frac{1}{20} = \frac{19}{20}
\]

7. What derivative does \( \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \) represent?

\[
\text{derivative of } f(x) = \sqrt{x} \text{ at } x = 4
\]

because \( \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} = \frac{1}{2} \frac{1}{\sqrt{4}} \]