In this section we compute average values of functions and discuss the Integral Mean Value Theorem.

Suppose you model the outside temperature as

\[ f(t) = 10 + 6 \sin\left(\frac{\pi t}{12}\right) \]

where \( t \) is measured in hours since 7am, and \( f \) is measured in Celsius.

What is the average temperature between 7am and 7pm?
To estimate the average temperature, we could measure the temperature at 9am, 11am, 1pm, 3pm, 5pm, and 7pm, and then take the average of those values:

\[
\bar{f} \approx \frac{f(2) + f(4) + f(6) + f(8) + f(10) + f(12)}{6}
\]

or more generally, if we measure \( n \) times:

\[
\bar{f} \approx \frac{\sum_{i=1}^{n} f(i\Delta x)}{n}
\]

where \( \Delta x = 12/n \)

**Definition.** The **average value** of a function \( f(x) \) over an interval \([a, b]\) is

\[
\bar{f} = \frac{1}{b - a} \int_{a}^{b} f(x) \, dx
\]

Idea: Let \( \Delta x = \frac{b - a}{n} \)

We can approximate the average value using \( n \) evenly spaced data points as

\[
\bar{f} \approx \frac{\sum_{i=1}^{n} f(a + i\Delta x)}{n}
\]

(average of \( n \) data points)

\[
= \frac{\sum_{i=1}^{n} f(a + i\Delta x)\Delta x}{b - a}
\]

(replace \( \frac{1}{n} \) by \( \frac{\Delta x}{b - a} \))

\[
= \frac{1}{b - a} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x
\]

If we take the limit as \( n \) goes to infinity, the Riemann sum becomes an integral, and we get

\[
\bar{f} = \frac{1}{b - a} \int_{a}^{b} f(x) \, dx
\]
Example. Suppose you model the temperature outside as

\[ f(t) = 10 + 6 \sin \left( \frac{\pi t}{12} \right) \]

where \( t \) is measured in hours since 7am, and \( f \) is measured in Celsius.

To compute the average temperature between 7am and 7pm, we compute

\[
\frac{1}{12} \int_0^{12} f(x) \, dx
\]

We do a u-substitution on the second part:

\[
u = \frac{\pi t}{12}, \quad du = \frac{\pi}{12} \, dt, \quad u(0) = 0, \quad u(12) = \pi
\]

\[
f_{\text{avg}} = 10 + \frac{1}{12} \int_0^\pi 6 \sin(u) \frac{12}{\pi} \, du
\]

\[
= 10 + \frac{6}{\pi}(-\cos u) \bigg|_0^\pi = 10 + \frac{6}{\pi}(2) \approx 13.8 \quad \text{(about 57°F)}
\]

Example. Find the average value of \( f(x) = x^2 \) on the interval \([-1, 2]\).

Try this yourself. Pause the video!

We use

\[
f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

\[
= \frac{1}{2 - (-1)} \int_{-1}^2 x^2 \, dx
\]

\[
= \frac{1}{3} \int_{-1}^2 x^2 \, dx = \frac{1}{3} \frac{1}{3} x^3 \bigg|_{-1}^2 = 1
\]
**Integral Mean Value Theorem.**

If $f$ is continuous on $[a, b]$, then there exists a number $c$ in $[a, b]$ so that

$$f(c) = f_{\text{avg}} = \frac{1}{b - a} \int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b - a)$$

For instance, the average value of $f(t) = 10 + 6 \sin\left(\frac{\pi t}{12}\right)$ on $[0, 12]$ was $f_{\text{avg}} = 10 + 12/\pi \approx 13.8$.

The theorem says that there must be at least one time between 7am and 7pm when the actual temperature is $f_{\text{avg}}$.

The theorem also says that for that $c$,

$$\int_0^{12} f(x)\,dx = f(c)(12 - 0).$$

Compare this theorem with the **Mean Value Theorem**:

Suppose that $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$. Then there exists a number $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$
Have a good winter break!