Please take a moment to just breathe.
We do two more examples of using integrals to find the total work done.
English Units:

A **pound** is a unit of force. It is abbreviated as “**lb**”.

A **foot** is about as long as a large human foot, or about 30.48 cm.

Work is measured in **foot-pounds**.

Metric Units:

Force is measured in **Newtons**, or **kg m/s^2**

Work is measure in **Newton-meters**, also called **joules**.
If a constant force of size $F$ acts to move an object a distance $D$ in some direction, then the work done is

$$W = FD$$

Lifting a 2 lb ball up 6 feet requires 12 ft-lbs of work:

$$FD = (2 \text{ lbs})(6 \text{ ft}) = 12 \text{ ft-lbs}.$$  

Lifting a 1 kg ball up 2 meters requires

$$FD = (9.8 \text{ m/s}^2)(1 \text{ kg})(2 \text{ m}) = 19.6 \text{ Newton-meters}$$

Holding a ball still in the air requires no work since the distance traveled is 0.
If the force is not constant, then we use integrals to compute the work done. The process is very similar to the process we used for computing volumes:

0. Attach a convenient coordinate system.

1. Split the problem up into small bits.

2. For each bit, determine how much work is done.

3. Find the limits of integration.

4. Compute the definite integral.
Example. A heavy rope, 50 ft long, weighs 1/2 pound per foot. It hangs over the edge of a 120 ft building. How much work does it take to pull the entire rope to the top?

0. Attach a coordinate system.

We let $y$ be the height off the ground, in ft.

1. Break the problem into bits.

How much work does it take to lift the small chunk of rope between $y$ and $y + \Delta y$ up to a height of 120 ft?
2. Compute the work to move a bit of rope.

Consider the bit of rope between $y$ and $y + \Delta y$.

We need to find

2a) the force acting on the bit
2b) the distance the bit moves.
2c) the work done to move the bit
Example. A heavy rope, 50 ft long, weighs 1/2 pound per foot. It hangs over the edge of building of 120 ft. How much work is done to lift it to the top?

2a) The force needed to counteract gravity is the same as the weight of the bit.

\[
\text{force}(y) = \frac{1}{2} \Delta y \text{ pounds.}
\]

2b) The bit travels from height \( y \) up to height 120, so

\[
\text{distance}(y) = 120 - y
\]
To compute the work, compute

\[ \Delta W = F(y)D(y) \]

\[ \text{force}(y) = \frac{1}{2} \Delta y \]

\[ \text{distance}(y) = 120 - y \]

\[ \Delta W = \frac{1}{2} \Delta y (120 - y) \]
2. We have found the work done for the small bit.

\[ \Delta W = \frac{1}{2} \Delta y (120 - y) \]

3. Find limits of integration.

When we start,
the highest bit of rope is at \( y = 120 \).
and the lowest bit of rope is at \( y = 70 \),
since the rope is only 50 feet long.

So the limits are \( y = 70 \) to \( y = 120 \)
4. We can finally solve an integral: the total work is

\[
\int_{70}^{120} \frac{1}{2} (120 - y) dy
\]

Do a u-substitution:

\[
u = 120 - y \quad u(120) = 0
\]

\[
du = -dy \quad u(70) = 50
\]

\[
\int_{50}^{0} \frac{1}{2} u (-du) = \int_{0}^{50} \frac{1}{2} u \ du = \frac{1}{4} u^2 \Big|_{50}^{0} = 625 \text{ ft-lbs.}
\]
Example. For comparison, let’s do the same problem using the coordinate system that the textbook uses, where the variable $u$ measures distance down from the top.

1. Break the problem into bits.

How much work does it take to lift the small chunk of rope between $u + \Delta u$ and up to a height of 120 ft?
We need to find

2a) the force acting on the bit
2b) the distance the bit moves.
2c) the work done to move the bit
2a) The force needed to counteract gravity is the same as the weight of the bit.

\[
\text{force}(u) = \frac{1}{2} \Delta u \quad \text{pounds.}
\]

2b) The bit moves from position \( u \) to the top.

\[
\text{distance}(u) = u
\]

2c. Total work to move the bit:

\[
\Delta W = \frac{1}{2} u \Delta u
\]
2. The work to move the small bit of rope is \[ \Delta W = \frac{1}{2} u \Delta u \]

3. Find limits of integration.

The lowest value for \( u \) is \( u = 0 \) and the highest value for \( u \) is \( u = 50 \) since the rope is only 50 m long.

So the limits are \( u = 0 \) to \( u = 50 \).

4. We compute the integral

\[
\int_{0}^{50} \frac{1}{2} u \, du = \frac{1}{4} u^2 \bigg|_{0}^{50} = 625
\]
Example. A leaky 10 kg bucket is lifted from the ground to a height of 12 meters at a constant speed, using a rope that weighs 0.8kg/m. Initially the bucket has 36 kg of water, but it leaks at a constant rate, finishing as the bucket reaches 12m. How much work was done?

0. Attach a coordinate system.

We let y be the height off the ground, in meters.

1. Break the problem into bits.

How much work does it take to lift the bucket from y to y + Δy?
2. find the work done to move from y to y + \(\Delta y\)

2a) Find the distance things move
2b) Find the mass being moved
2c) Find the force needed
2d) Find the work done
2a) Find the distance things move

We are moving everything a tiny bit in the y direction:

\[ \text{distance}(y) = \Delta y \]
Example. A leaky 10 kg bucket is lifted from the ground to a height of 12 meters at a constant speed, using a rope that weighs 0.8kg/m. Initially the bucket has 36 kg of water, but it leaks at a constant rate, finishing as the bucket reaches 12m. How much work was done?

2b) Find the mass being lifted at y

Mass of bucket: 10 kg
Mass of rope still hanging off: 0.8(12-y)
Mass of water:
  - when y = 0, water is 36 kg
  - when y = 12, water is 0 kg
Constant drip means linear function
Mass of water = 36 - 3y

So mass(y) = 10 + 0.8(12-y) + 36-3y kg
2c) Find the force needed: we have to pull against gravity:

\[
\text{mass}(y) = 10 + 0.8(12-y) + 36-3y
\]

\[
\text{Force}(y) = 9.8 \text{ mass}(y)
\]

2d) Find the work done for the bit.

\[
\text{Work} = \text{force} \times \text{distance}
\]

\[
\Delta W = 9.8 \left( 10 + 0.8(12 - y) + 36 - 3y \right) \Delta y
\]

\[
= 9.8(55.6 - 3.8y)\Delta y
\]
3. Find the limits of integration:

Start when the bucket is at \( y = 0 \)
End when the bucket is at \( y = 12 \).

4. Total Work done:

\[
\int_{0}^{12} 9.8(55.6 - 3.8y) \, dy \approx 3860 \text{ N}
\]