Math 221
Week 14 part 1

Volumes of Revolution: method of shells
Please take a moment to just breathe.
We discuss the method of shells to compute volumes of solids of revolutions.
We will follow a similar procedure, but use thin cylinders instead of flat slices.

1. Chop object into thin cylinders.

2. Find the volume of each cylinder.

3. Add up the bits of volume to get a Riemann sum.

4. Solve the resulting definite integral.
Example: find the volume of the solid of revolution obtained by rotating the the region between the curves $y = x$ and $y = x^2$ about the y-axis. Last time we used washers:

\[
\Delta V = \left( \pi R(y)^2 - \pi r(y)^2 \right) \Delta y
\]

\[
V = \int_a^b \pi R(y)^2 - \pi r(y)^2 \, dy
\]
This time we’ll chop the region into concentric cylinders. Think of the cylinders as nested toilet paper rolls of different sizes. Each one has its own height and radius.
We need to find the height, radius, and thickness of the cylinder at position $x$.

The thickness will be $\Delta x$ since it is a small change in the $x$ direction.
For the cylinder at position $x$:
The height is $H(x) = x - x^2$.
The radius is $R(x) = x$. 
To find the volume of a cylinder with height $H$, radius $R$, and thickness $\Delta x$, we “unroll” it into a flat slab:

The total volume of one cylinder is $\Delta V = 2\pi R H \Delta x$. 

$\Delta x$ 

$R$ 

$circumference = 2\pi R$ 

$H$ 

$\Delta x$ 

$H$ 

The total volume of one cylinder is $\Delta V = 2\pi R H \Delta x$. 

\[ H(x) = x - x^2 \]
\[ R(x) = x \]
\[ \Delta V = 2\pi RH \Delta x = 2\pi x(x - x^2) \Delta x \]

Volume of Solid \[ = \int_0^1 2\pi R(x)H(x)dx = \int_0^1 2\pi x(x - x^2)dx \]
Finally, we compute the integral:

\[
\int_0^1 2\pi R(x)H(x)\,dx = \int_0^1 2\pi x(x - x^2)\,dx
\]

\[
= 2\pi \int_0^1 x^2 - x^3 \,dx
\]

\[
= 2\pi \left( \frac{1}{3}x^3 - \frac{1}{4}x^4 \right) \bigg|_0^1
\]

\[
= \frac{2}{12}\pi = \frac{1}{6}\pi.
\]
Here’s the same problem using washers, for comparison:

Volume = \int_0^1 \pi R(y)^2 - \pi r(y)^2 \, dy = \int_0^1 \pi (\sqrt{y})^2 - \pi (y)^2 \, dy

= \pi \left( \frac{1}{2} y^2 - \frac{1}{3} y^3 \right) \bigg|_0^1 = \frac{1}{6} \pi
Example: find the volume of the solid of revolution obtained by rotating the region between the curves $y = x$ and $y = x^2$ about the line $x = 2$.

Try this one yourself, but using shells instead of washers. (Please pause the video!)
Example: find the volume of the solid of revolution obtained by rotating the region between the curves $y = x$ and $y = x^2$ about the line $x = 2$.

$H(x) = x - x^2$

$R(x) = 2 - x$
Example: find the volume of the solid of revolution obtained by rotating the region between the curves $y = x$ and $y = x^2$ about the line $x = 2$.

$$H(x) = x - x^2$$
$$R(x) = 2 - x$$

Volume = $\int_{0}^{1} 2\pi R(x)H(x) \, dx$

$$= \int_{0}^{1} 2\pi (2 - x)(x - x^2) \, dx = 2\pi \int_{0}^{1} 2x - 3x + x^3 \, dx$$

$$= 2\pi \left( x^2 - x^3 + x^4/4 \right) \bigg|_{0}^{1} = \pi/2.$$
Here’s a summary of the method for solids of revolution.

1. Decide if you’d rather do a dx or a dy integral.

The “thin” dimension will be dx if it is a bit of change in the x-direction, or dy if it is a bit of change in the y-direction.

2. Determine what shape each piece is: disk, washer, shell, or other.

3. Find the volume of each piece.

4. Find the limits of integration.

5. Compute the definite integral.
For rotating about a horizontal axis (like the x-axis):

**Disks:** $\int_{a}^{b} \pi R(x)^2 \, dx$  
**Washers:** $\int_{a}^{b} \pi (R(x)^2 - r(x)^2) \, dx$

**Shells:** $\int_{c}^{d} 2\pi R(y)H(y) \, dy$

For rotating about a vertical axis (like the y-axis):

**Disks:** $\int_{c}^{d} \pi R(y)^2 \, dy$  
**Washers:** $\int_{c}^{d} \pi (R(y)^2 - r(y)^2) \, dy$

**Shells:** $\int_{a}^{b} 2\pi R(x)H(x) \, dx$