Math 221
Week 13 part 3

Other shapes
Please take a moment to just breathe.
In this section we chop other solids into slices.
We will follow the same procedure:

1. Chop object into thin slices.
2. Find the volume of each slice.
3. Add up the bits of volume to get a Riemann sum.
4. Solve the resulting definite integral.
Example. A solid $S$ is made with a triangular base with vertices $(0,0)$, $(0,3)$, and $(3,0)$.

The cross sections perpendicular to the $x$-axis are equilateral triangles. Find the volume.
The triangle at position $x$ has base length $B = 3 - x$. 

\[(0,3) \quad y = 3-x \quad (3,0) \] 

\[B = 3-x\]
The triangle at position $x$ has base length $B = 3 - x$.

The height of an equilateral triangle is

$$H = \frac{\sqrt{3}}{2}B.$$ 

The area of the triangle is

$$A = \frac{BH}{2} = \frac{\sqrt{3}}{4}B^2.$$ 

$$A(x) = \frac{\sqrt{3}}{4}(3 - x)^2$$
Think of chopping the solid into thin slices.

Each slice has thickness $\Delta x$.

The slice at $x$ has volume

$$\Delta V = A(x) \Delta x$$

$$\Delta V = \frac{\sqrt{3}}{4}(3 - x)^2 \Delta x$$

So the total volume is

$$V = \int_0^3 \frac{\sqrt{3}}{4}(3 - x)^2 \, dx$$
We compute the definite integral:

\[ V = \int_{0}^{3} \frac{\sqrt{3}}{4} (3 - x)^2 \, dx \]

We can use the method of substitution:

\[ u = 3 - x \quad u(0) = 3 \]
\[ du = - \, dx \quad u(3) = 0 \]

\[ V = \int_{3}^{0} \frac{\sqrt{3}}{4} u^2 (-1) \, du = \int_{0}^{3} \frac{\sqrt{3}}{4} u^2 \, du = \frac{\sqrt{3}}{4} \frac{u^3}{3} \bigg|_{0}^{3} = \frac{9}{4} \sqrt{3} \]
Example. The Great Pyramid Khufu has a square base with sides 750 ft and height 450 ft. Find its volume by adding up the volume of its horizontal slices.

Please try this yourself. (Pause the video.)
Example. The Great Pyramid Khufu has a square base with sides 750 ft and height 450 ft. Find its volume by adding up the volume of its horizontal slices.

Before we get started, here’s a formula for any pyramid:

\[ V = \frac{1}{3}BH \]

where

\( B \) is the area of the base, and

\( H \) is the height,

so we expect to get

\[ V = \frac{1}{3}(750)(750)(450) \]
Example. The Great Pyramid Khufu has a square base with sides 750 ft and height 450 ft. Find its volume by adding up the volume of its horizontal slices.

Each slice has a square base.

How big is the square at height $y$?
We use similar triangles to find the edge length of the square at height $y$: 

$$
\frac{(450-y) \cdot 750}{450}
$$
The edge length and area of the square at height $y$ are

$$E(y) = 750 - \frac{5}{3}y$$

$$A(y) = (750 - \frac{5}{3}y)^2$$
The area of the square at height \( y \) is

\[
A(y) = \left(750 - \frac{5}{3}y\right)^2
\]

The width of the slice is \( \Delta y \)

The volume of the slice is

\[
\Delta V = A(y)\Delta y = \left(750 - \frac{5}{3}y\right)^2\Delta y
\]

The total volume is

\[
\int_{0}^{450} A(y)\,dy = \int_{0}^{450} \left(750 - \frac{5}{3}y\right)^2\,dy
\]
\[
\int_{0}^{450} A(y)\,dy = \int_{0}^{450} (750 - \frac{5}{3}y)^2 \,dy
\]

Do a substitution:

\[
u(y) = 750 - \frac{5}{3}y \quad \quad u(0) = 750
\]

\[
du = -\frac{5}{3} \,dy \quad \quad u(450) = 0
\]

\[
\int_{0}^{750} u^2 \left( -\frac{3}{5} \right) du = \frac{3}{5} \int_{0}^{750} u^2 \,du = \frac{3}{5} \frac{1}{3} u^3 \bigg|_{0}^{750}
\]

\[
= \frac{1}{3} (750)(750)(450), \quad \text{as we expected.}
\]