Volumes of Revolution: rotating about other axes

We use integrals to compute volumes of solids of revolution when the axis is any horizontal or vertical line in the plane.

We will follow the same procedure:

1. Chop object into thin slices.

2. Find the volume of each slice.

3. Add up the bits of volume to get a Riemann sum.

4. Solve the resulting definite integral.
Suppose we rotate a region about the y-axis.

How do we compute the volume of the resulting solid?

To compute the volume, we chop up the solid into thin slices and measure the volume of each slice.

For instance, we could look at the slice at position $y_i$.

The volume of this slice will be equal to the area of the top of the slice times the thickness of the slice.

Next we add up all the bits of volume.

The total volume will be approximately

$$\sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} A(y_i) \Delta y$$

This is a Riemann sum. If we let $n$ go to infinity, we get

$$V = \int_{y=a}^{y=b} A(y) \, dy = \int_{a}^{b} \pi [f(y)]^2 \, dy$$

where the slices go from $y = a$ to $y = b$. 
Example: find the volume of the solid of revolution obtained by rotating the region between the curves $y = x$ and $y = x^2$ about the y-axis.

Please try this yourself. (Pause the video.)

Guide questions for setting up the integral:
1. Are we doing a dx or dy integral?
2. Do we want washers, disks, or something else?
3. What is the area of a face of a slice?
4. What are the limits of integration?

Example: find the volume of the solid of revolution obtained by rotating the region between the curves $y = x$ and $y = x^2$ about the y-axis. We’ll need washers:

Outer radius of washer: $R(y) = \sqrt{y}$
Inner radius of washer: $r(y) = y$

Finally, we compute the integral:

$$\int_0^1 \pi(\sqrt{y})^2 - \pi y^2 \, dy = \pi \int_0^1 y - y^2 \, dy = \pi \left( \frac{1}{2} y^2 - \frac{1}{3} y^3 \right) \bigg|_0^1 = \frac{1}{6} \pi$$
One more variation: rotating around $x = a$ or $y = a$

Example: find the volume of the solid of revolution obtained by rotating the region between the curves $y = x$ and $y = x^2$ about the line $x = 2$.

To find the outer radius at $y$, we need to find the distance between the axis of rotation and the outer edge of the washer at height $y$.

The two endpoints of the segment are $(y, y)$ and $(2, y)$, which are distance $R(y) = 2 - y$ apart.

Example: find the volume of the solid of revolution obtained by rotating the region between the curves $y = x$ and $y = x^2$ about the line $x = 2$.

We'll use washers again, with thickness $\Delta y$.

To find the inner radius at $y$, we need to find the distance between the axis of rotation and the inner edge of the washer at $y$.

The two endpoints are $(\sqrt{y}, y)$ and $(2, y)$, which are distance $r(y) = 2 - \sqrt{y}$ apart.
Example: find the volume of the solid of revolution obtained by rotating the region between the curves \( y = x \) and \( y = x^2 \) about the line \( x = 2 \).

\[
R(y) = 2 - y \\
r(y) = 2 - \sqrt{y}
\]

Volume = \( \int_0^1 \pi R(y)^2 - \pi r(y)^2 \, dy \)

\[
\pi \int_0^1 (2 - y)^2 - (2 - \sqrt{y})^2 \, dy
\]

Finally, we compute the definite integral:

\[
\pi \int_0^1 (2 - y)^2 - (2 - \sqrt{y})^2 \, dy \\
= \pi \int_0^1 y^2 - 5y + 4\sqrt{y} \, dy \\
= \pi \left( \frac{1}{3}y^3 - \frac{5}{2}y^2 + 4\frac{2}{3}y^{3/2} \right) \bigg|_0^1 = \frac{\pi}{2}
\]