Math 221
Week 13 part 1

Volumes of Revolution:
Method of Disks and Washers

Please take a moment to just breathe.

We use integrals to compute volumes of solids of revolution.

All of the problems for this week will follow the same procedure:

1. Chop object into thin slices.
2. Find the volume of each slice.
3. Add up the bits of volume to get a Riemann sum.
4. Solve the resulting definite integral.
Suppose we are given a region in the plane, such as in the figure on the left.

We can form a solid of revolution by rotating the region about an axis of revolution.

How do we compute the volume of the new solid?

Example: this region lies under graph of \( y = f(x) \) between \( x = a \) and \( x = b \).

To compute the volume, we chop up the solid into thin slices and measure the volume of each slice.

For instance, we could look at the slice at position \( x_i \).

The volume of this slice will be equal to the area of the front of the slice times the thickness of the slice.

The thickness of the slice is \( \Delta x \), since it is a small distance along the x-axis.

The radius of the slice is \( f(x_i) \).

The area of the front of the slice is \( A(x_i) = \pi[f(x_i)]^2 \).

So the volume of the slice at \( x_i \) is \( \Delta V_i = A(x_i)\Delta x \) or \( \Delta V_i = \pi[f(x_i)]^2\Delta x \).

Next we add up all the bits of volume.

The total volume will be approximately

\[
\sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} A(x_i)\Delta x
\]

This is a Riemann sum. If we let \( n \) go to infinity, we get

\[
V = \int_{a}^{b} A(x)dx = \int_{a}^{b} \pi[f(x)]^2 \, dx
\]

where the slices go from \( x = a \) to \( x = b \).
Example. Compute the volume of a sphere of radius 10.

We use the region under the graph of $y = \sqrt{100 - x^2}$ between $x = -10$ and $x = 10$.

We'll rotate this region around the x-axis.
Please try this problem yourself. (Pause the video!)

Radius of slice at $x_i$ is $\sqrt{100 - x_i^2}$
Area at $x_i$ is $A(x_i) = \pi \left( \sqrt{100 - x_i^2} \right)^2 = \pi(100 - x_i^2)$
Volume is $\Delta V_i = A(x_i)\Delta x = \pi(100 - x_i^2)\Delta x$

Finally, we compute the integral using the Fundamental Theorem of Calculus:

By symmetry,
$$\int_{-10}^{10} \pi(100 - x^2) \, dx = 2\int_{0}^{10} \pi(100 - x^2) \, dx$$
$$= 2\pi \left( 100x - \frac{1}{3}x^3 \right) \bigg|_{0}^{10}$$
$$= \frac{4}{3}(1000)\pi$$
Some variations on the theme:

1. Sometimes the slice looks like a washer instead of a disk:

   For instance, the region could be the area between the curves $y = f(x)$ and $y = g(x)$.

   ![Diagram](image)

In this case, we need to compute

inner radius $r(x_i)$
outer radius $R(x_i)$

Area:

$$A(x_i) = \pi R(x_i)^2 - \pi r(x_i)^2$$

Area of washer = area of big circle - area of small circle

$$= \pi R^2 - \pi r^2$$

If the top curve corresponds to the outer circle, and the bottom curve corresponds to the inner circle, then we get

$R(x) = f(x)$
$r(x) = g(x)$

and the total volume is

$$\int_a^b \pi R(x)^2 - \pi r(x)^2 \, dx = \int_a^b \pi f(x)^2 - \pi g(x)^2 \, dx$$

Example: find the volume of the solid of revolution obtained by rotating the region between the curves $y = x$ and $y = x^2$ about the x-axis.

Please try this yourself. (Pause the video.)
Example: find the volume of the solid of revolution obtained by rotating the region between the curves \( y = x \) and \( y = x^2 \) about the x-axis.

The graph will be split up into washers.

Outer radius: 
\[ R(x) = x \]

Inner radius: 
\[ r(x) = x^2 \]

Integral:
\[
\int_0^1 \pi R(x)^2 - \pi r(x)^2 \, dx = \int_0^1 \pi x^2 - \pi x^4 \, dx
\]

Example: find the volume of the solid of revolution obtained by rotating the region between the curves \( y = x \) and \( y = x^2 \) about the x-axis.

Finally, we compute the integral:
\[
\pi \int_0^1 x^2 - x^4 \, dx = \pi \left( \frac{1}{3} x^3 - \frac{1}{5} x^5 \right) \bigg|_0^1 = \frac{2\pi}{15}
\]