Math 221
Week 13 part 1

Volumes of Revolution:
Method of Disks and Washers
Please take a moment to just breathe.
We use integrals to compute volumes of solids of revolution.
All of the problems for this week will follow the same procedure:

1. Chop object into thin slices.

2. Find the volume of each slice.

3. Add up the bits of volume to get a Riemann sum.

4. Solve the resulting definite integral.
Suppose we are given a region in the plane, such as in the figure on the left.

We can form a solid of revolution by rotating the region about an axis of revolution.

How do we compute the volume of the new solid?

Example: this region lies under graph of \( y = f(x) \) between \( x = a \) and \( x = b \)
To compute the volume, we chop up the solid into thin slices and measure the volume of each slice.

For instance, we could look at the slice at position $x_i$.

The volume of this slice will be equal to the area of the front of the slice times the thickness of the slice.
The thickness of the slice is $\Delta x$, since it is a small distance along the x-axis.

The radius of the slice is $f(x_i)$.

The area of the front of the slice is $A(x_i) = \pi [f(x_i)]^2$

So the volume of the slice at $x_i$ is $\Delta V_i = A(x_i)\Delta x$ or $\Delta V_i = \pi [f(x_i)]^2 \Delta x$
Next we add up all the bits of volume.

The total volume will be approximately

\[ \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} A(x_i) \Delta x \]

This is a Riemann sum. If we let \( n \) go to infinity, we get

\[ V = \int_{a}^{b} A(x) dx = \int_{a}^{b} \pi [f(x)]^2 dx \]

where the slices go from \( x = a \) to \( x = b \).
Example. Compute the volume of a sphere of radius 10.

We use the region under the graph of \( y = \sqrt{100 - x^2} \) between \( x = -10 \) and \( x = 10 \).

We’ll rotate this region around the x-axis. Please try this problem yourself. (Pause the video!)
Example. Compute the volume of a sphere of radius 10.

Radius of slice at $x_i$ is $\sqrt{100 - x_i^2}$

Area at $x_i$ is $A(x_i) = \pi \left( \sqrt{100 - x_i^2} \right)^2 = \pi(100 - x_i^2)$

Volume is $\Delta V_i = A(x_i) \Delta x = \pi(100 - x_i^2) \Delta x$
Example. Compute the volume of a sphere of radius 10.

Total volume:

\[ V = \int_{-10}^{10} A(x) \, dx = \int_{-10}^{10} \pi(100 - x^2) \, dx \]
Finally, we compute the integral using the Fundamental Theorem of Calculus:

By symmetry,

\[ \int_{-10}^{10} \pi(100 - x^2) \, dx = 2 \int_{0}^{10} \pi(100 - x^2) \, dx \]

\[ = 2\pi \left( 100x - \frac{1}{3}x^3 \right) \bigg|_{0}^{10} \]

\[ = \frac{4}{3}(1000)\pi \]
Some variations on the theme:

1. Sometimes the slice looks like a washer instead of a disk:

For instance, the region could be the area between the curves $y = f(x)$ and $y = g(x)$.
In this case, we need to compute
inner radius $r(x_i)$
outer radius $R(x_i)$

Area:
$$A(x_i) = \pi R(x_i)^2 - \pi r(x_i)^2$$

Area of washer = 
area of big circle - area of small circle
$$= \pi R^2 - \pi r^2$$
If the top curve corresponds to the outer circle, and the bottom curve corresponds to the inner circle, then we get

\[ R(x) = f(x) \]
\[ r(x) = g(x) \]

and the total volume is

\[
\int_{a}^{b} \pi R(x)^2 - \pi r(x)^2 \, dx = \int_{a}^{b} \pi f(x)^2 - \pi g(x)^2 \, dx
\]
Example: find the volume of the solid of revolution obtained by rotating the region between the curves \( y = x \) and \( y = x^2 \) about the x-axis.

Please try this yourself. (Pause the video.)
Example: find the volume of the solid of revolution obtained by rotating the region between the curves $y = x$ and $y = x^2$ about the x-axis.
Example: find the volume of the solid of revolution obtained by rotating the region between the curves $y = x$ and $y = x^2$ about the x-axis.

The graph will be split up into washers.

Outer radius:
$R(x) = x$

Inner radius:
$r(x) = x^2$

Integral:
$$\int_0^1 \pi R(x)^2 - \pi r(x)^2 \, dx = \int_0^1 \pi x^2 - \pi x^4 \, dx$$
Example: find the volume of the solid of revolution obtained by rotating the region between the curves \( y = x \) and \( y = x^2 \) about the x-axis.

Finally, we compute the integral:

\[
\pi \int_0^1 x^2 - x^4 \, dx = \pi \left( \frac{1}{3} x^3 - \frac{1}{5} x^5 \right) \bigg|_0^1 = \frac{2\pi}{15}
\]