In this section we describe the Method of Substitution.

Recall the Chain Rule:

\[
[f(u(x))]' = f'(u(x))u'(x)
\]

The method of substitution will “undo” the Chain Rule:

\[
\int f'(u(x))u'(x)dx = f(u(x)) + C
\]

As an intermediate step, we will write the integral as

\[
\int f(u)du, \text{ where } du = u'(x)dx
\]
Example

\[(x^3 + 1)^{10} = 10(x^3 + 1)^9(3x^2)\]
so

\[
\int 10(x^3 + 1)^9(3x^2) \, dx = (x^3 + 1)^{10} + C
\]

Method of Substitution

Step 1. Look for a good “inner” function to use for \(u(x)\).

- Look for something inside parentheses or radicals or in the denominator.
- Look for something whose derivative is available.

\[
\int 30x^2(x^3 + 1)^9 \, dx = (x^3 + 1)^{10} + C
\]

\(u(x) = x^3 + 1\) is a good choice since it’s in parentheses and there’s an \(x^2\) outside.

Step 2. Compute “du”:

\(u = x^3 + 1\)

\[
\frac{du}{dx} = 3x^2
\]

\(du = 3x^2 \, dx\) yes, this is legal notation.

Step 3. Rewrite the \(dx\) integral as a \(d\) integral.

\(u = x^3 + 1\)

\(du = 3x^2 \, dx\)

\[
\int 30x^2(x^3 + 1)^9 \, dx \text{ can be written in many different ways:}
\]

\[
\int 10(x^3 + 1)^9 \, 3x^2 \, dx = \int 10u^9 \, du
\]
Step 3. Rewrite the $\int dx$ integral as a $\int du$ integral.

$u = x^3 + 1$
$du = 3x^2dx$

$\int 30x^2(x^3 + 1)^9dx$ can be written in many different ways:

$\int 30u^9 x^2 dx = \int 30u^9 \frac{1}{3} du = \int 10u^9 du$

Here are a few for you to try. (Please pause the video!)

$\int 5x^4 \sin(x^5)dx$
$\int x^4 \sin(x^5)dx$
$\int \frac{x^2}{x^3 + 1} dx$
$\int \frac{\cos 3x}{\sqrt{\sin 3x}} dx$

Step 4. Solve the $\int du$ integral

$\int 10u^9 du = u^{10} + C$

Step 5. Replace $u$ with the function of $x$:

$u^{10} + C = (x^3 + 1)^{10} + C$

Answers:

$\int 5x^4 \sin(x^5)dx = -\cos(x^5) + C$

$\int x^4 \sin(x^5)dx = -\frac{1}{5} \cos(x^5) + C$

$\int \frac{x^2}{x^3 + 1} dx = \frac{1}{3} \ln |x^3 + 1| + C$

$\int \frac{\cos 3x}{\sqrt{\sin 3x}} dx = \frac{2}{3} (\sin 3x)^{1/2} + C$
\int 5x^4 \sin(x^5) dx

u(x) = x^5
\quad du = 5x^4 dx

\int 5x^4 \sin(x^5) dx = \int \sin(x^5) \cdot 5x^4 dx = \int \sin(u) du

\int \sin(u) du = -\cos(u) + C = -\cos(x^5) + C

\int x^2 dx

u(x) = x^3 + 1
\quad du = 3x^2 dx

\int x^2 dx = \int \frac{1}{x^3 + 1} x^2 dx = \int \frac{1}{x^3 + 1} \cdot 1 \cdot 3 du

\int \frac{1}{u} du = \frac{1}{3} \ln |u| + C = \frac{1}{3} \ln |x^3 + 1| + C

\int \frac{\cos 3x}{\sqrt{\sin 3x}} dx

u(x) = \sin(3x)
\quad du = 3 \cos(3x) dx

\int \frac{\cos 3x}{\sqrt{\sin 3x}} dx = \int (\sin 3x)^{-1/2} \cos 3x dx = \int u^{-1/2} \frac{1}{3} du

\int \frac{1}{u} du = \frac{1}{3} u^{1/2} + C = \frac{2}{3} (\sin 3x)^{1/2} + C
These integrals require a bit more guesswork. (Please try these yourself. Answers on the next page, full solutions after.)

\[
\int \frac{4}{x(\ln x + 1)^2} \, dx \\
\int \frac{x}{\sqrt{1 - x^4}} \, dx \\
\int \frac{e^x}{1 + e^{2x}} \, dx \\
\int x \sqrt{2 - x} \, dx
\]

Here are the answers so you can check your work:

\[
\int \frac{4}{x(\ln x + 1)^2} \, dx = \frac{-4}{\ln x + 1} + C \\
\int \frac{x}{\sqrt{1 - x^4}} \, dx = \frac{1}{2} \arcsin(x^2) + C \\
\int \frac{e^x}{1 + e^{2x}} \, dx = \arctan(e^x) + C \\
\int x \sqrt{2 - x} \, dx = \frac{-4}{3} (2 - x)^{3/2} + \frac{2}{5} (2 - x)^{5/2} + C
\]

Here, we look for something inside parentheses that has the derivative available. Since \(1/x\) is available to be a derivative, let's try

\[
u(x) = \ln x + 1 \\
\text{du} = 1/x \, dx.
\]

\[
\int \frac{4}{x(\ln x + 1)^2} \, dx = \int \frac{4}{(\ln x + 1)^2} \cdot \frac{dx}{x} = \int \frac{4}{u^2} \, du
\]

\[
4u^{-2} \, du = -4u^{-1} + C = -4 \left( \frac{1}{\ln x + 1} \right) + C
\]

Here, the first thing we might try is \(u(x) = 1 - x^4\). However, there's no factor of \(x^3\) available for the derivative.

We do have a factor of \(x\) available, so next we try

\[
u(x) = x^2 \\
\text{du} = 2x \, dx
\]

\[
\int \frac{x \, dx}{\sqrt{1 - x^4}} = \int \frac{1}{2} \cdot \frac{du}{\sqrt{1 - u^2}} = \frac{1}{2} \arcsin(u) + C
\]

\[
= \frac{1}{2} \arcsin(x^2) + C
\]
\[ \int \frac{e^x}{1 + e^{2x}} \, dx \]

Here, we might try \( u(x) = 1 + e^{2x} \). However, there’s no \( e^{2x} \) available to be the derivative. However, there is an \( e^x \) available, so try

\[ u(x) = e^x \]
\[ du = e^x \, dx \]

\[ \int \frac{e^x \, dx}{1 + e^{2x}} = \int \frac{du}{1 + u^2} = \arctan u + C \]
\[ = \arctan(e^x) + C \]

\[ \int x\sqrt{2 - x} \, dx \]

Here, take \( u(x) \) to be what’s under the radical:

\[ u(x) = 2 - x \quad \text{and} \quad du = - \, dx \]

To do the substitution, we also need to write \( x \) in terms of \( u \):

\[ x = 2 - u \]

\[ \int x\sqrt{2 - x} \, dx = \int (2 - u)\sqrt{u} \, (-du) = \int (-2u^{1/2} + u^{3/2}) \, du \]
\[ -2 \left( \frac{2}{3} \right) u^{3/2} + (2/5)u^{5/2} + C \]
\[ = - \left( \frac{4}{3} \right)(2 - x)^{3/2} + (2/5)(2 - x)^{5/2} + C \]