Math 221
Week 12 part 1
Method of Substitution
Please take a moment to just breathe.
In this section we describe the Method of Substitution.
Recall the Chain Rule:

\[ [f(u(x))]' = f'(u(x))u'(x) \]

The method of substitution will “undo” the Chain Rule:

\[ \int f'(u(x))u'(x)dx = f(u(x)) + C \]

As an intermediate step, we will write the integral as

\[ \int f(u)du, \quad \text{where} \quad du = u'(x)dx \]
Example

\[ (x^3 + 1)^{10} \] \[= 10(x^3 + 1)^9(3x^2) \]

so

\[ \int 10(x^3 + 1)^9(3x^2) \, dx = (x^3 + 1)^{10} + C \]
Method of Substitution

Step 1. Look for a good “inner” function to use for $u(x)$.

- Look for something inside parentheses or radicals or in the denominator.
- Look for something whose derivative is available.

\[
\int 30x^2(x^3 + 1)^9 \, dx = (x^3 + 1)^{10} + C
\]

$u(x) = x^3 + 1$ is a good choice since it’s in parentheses and there’s an $x^2$ outside.
Step 2. Compute “du”:

\[ u = x^3 + 1 \]

\[ \frac{du}{dx} = 3x^2 \]

\[ du = 3x^2\,dx \]

yes, this is legal notation.
Step 3. Rewrite the \( \int dx \) integral as a \( \int du \) integral.

\[
\begin{align*}
\quad u &= x^3 + 1 \\
du &= 3x^2dx
\end{align*}
\]

\[
\int 30x^2(x^3 + 1)^9dx \quad \text{can be written in many different ways:}
\]

\[
\int 10(x^3 + 1)^9 \quad 3x^2 \quad dx = \int 10u^9 \quad du
\]

\[
10u^9 \\
du
\]
Step 3. Rewrite the \( \int dx \) integral as a \( \int du \) integral.

\[ u = x^3 + 1 \]
\[ du = 3x^2dx \]

\[ \int 30x^2(x^3 + 1)^9 dx \] can be written in many different ways:

\[ \int 30(x^3 + 1)^9 x^2 dx = \int 30u^9 \frac{1}{3} du = \int 10u^9 du \]
Step 4. Solve the \( \int du \) integral

\[
\int 10u^9 \, du = u^{10} + C
\]

Step 5. Replace \( u \) with the function of \( x \):

\[
u^{10} + C = (x^3 + 1)^{10} + C
\]
Here are a few for you to try. (Please pause the video!)

\[ \int 5x^4 \sin(x^5) \, dx \]
\[ \int x^4 \sin(x^5) \, dx \]
\[ \int \frac{x^2}{x^3 + 1} \, dx \]
\[ \int \frac{\cos 3x}{\sqrt{\sin 3x}} \, dx \]
Answers:

\[ \int 5x^4 \sin(x^5) \, dx = - \cos(x^5) + C \]

\[ \int x^4 \sin(x^5) \, dx = - \frac{1}{5} \cos(x^5) + C \]

\[ \int \frac{x^2}{x^3 + 1} \, dx = \frac{1}{3} \ln |x^3 + 1| + C \]

\[ \int \frac{\cos 3x}{\sqrt{\sin 3x}} \, dx = \frac{2}{3} (\sin 3x)^{1/2} + C \]
\[
\int 5x^4 \sin(x^5) \, dx
\]

\[u(x) = x^5\]
\[du = 5x^4 \, dx\]

\[
\int 5x^4 \sin(x^5) \, dx = \int \sin(x^5) \cdot 5x^4 \, dx = \int \sin(u) \, du
\]

\[
\int \sin(u) \, du = - \cos(u) + C = - \cos(x^5) + C
\]
\[
\int x^4 \sin(x^5) \, dx
\]

\[u(x) = x^5\]
\[du = 5x^4 \, dx\]

\[
\int x^4 \sin(x^5) \, dx = \int \sin(x^5) \, x^4 \, dx = \int \sin(u) \frac{1}{5} \, du
\]

\[
\int \sin(u) \frac{1}{5} \, du = -\frac{1}{5} \cos(u) + C = -\frac{1}{5} \cos(x^5) + C
\]
\[ \int \frac{x^2}{x^3 + 1} \, dx \]

\[ u(x) = x^3 + 1 \]
\[ du = 3x^2 \, dx \]

\[ \int \frac{x^2}{x^3 + 1} \, dx = \int \frac{1}{x^3 + 1} \, x^2 \, dx = \int \frac{1}{u} \, \frac{1}{3} \, du \]

\[ \int \frac{1}{u} \, du = \frac{1}{3} \ln |u| + C = \frac{1}{3} \ln |x^3 + 1| + C \]
\[ \int \frac{\cos 3x}{\sqrt{\sin 3x}} \, dx \]

\[ u(x) = \sin(3x) \]
\[ du = 3 \cos(3x) \, dx \]

\[ \int \frac{\cos 3x}{\sqrt{\sin 3x}} \, dx = \int (\sin 3x)^{-1/2} \cos 3x \, dx = \int u^{-1/2} \frac{1}{3} \, du \]

\[ \int \frac{1}{3} u^{-1/2} \, du = \frac{2}{3} u^{1/2} + C = \frac{2}{3} (\sin 3x)^{1/2} + C \]
These integrals require a bit more guesswork. (Please try these yourself. Answers on the next page, full solutions after.)

\[
\int \frac{4}{x(\ln x + 1)^2} \, dx
\]

\[
\int \frac{x}{\sqrt{1 - x^4}} \, dx
\]

\[
\int \frac{e^x}{1 + e^{2x}} \, dx
\]

\[
\int x\sqrt{2 - x} \, dx
\]
Here are the answers so you can check your work:

\[
\int \frac{4}{x(\ln x + 1)^2} \, dx = -\frac{4}{\ln x + 1} + C
\]

\[
\int \frac{x}{\sqrt{1 - x^4}} \, dx = \frac{1}{2} \arcsin(x^2) + C
\]

\[
\int \frac{e^x}{1 + e^{2x}} \, dx = \arctan(e^x) + C
\]

\[
\int x\sqrt{2 - x} \, dx = \frac{-4}{3}(2 - x)^{3/2} + \frac{2}{5}(2 - x)^{5/2} + C
\]
\[
\int \frac{4}{x(\ln x + 1)^2} \, dx
\]

Here, we look for something inside parentheses that has the derivative available. Since \(1/x\) is available to be a derivative, let's try

\[
u(x) = \ln x + 1
\]

\[
du = \frac{1}{x} \, dx.
\]

\[
\int \frac{4}{x(\ln x + 1)^2} \, dx = \int \frac{4}{(\ln x + 1)^2} \frac{dx}{x} = \int \frac{4}{u^2} \, du
\]

\[
\int 4u^{-2} \, du = -4u^{-1} + C = \frac{-4}{(\ln x + 1)} + C
\]
Here, the first thing we might try is \( u(x) = 1 - x^4 \). However, there’s no factor of \( x^3 \) available for the derivative.

We do have a factor of \( x \) available, so next we try

\[
\begin{align*}
u(x) &= x^2 \\
du &= 2x \, dx
\end{align*}
\]

\[
\int \frac{x \, dx}{\sqrt{1 - x^4}} = \int \frac{\frac{1}{2} \, du}{\sqrt{1 - u^2}} = \frac{1}{2} \arcsin(u) + C
\]

\[
= \frac{1}{2} \arcsin(x^2) + C
\]
\[ \int \frac{e^x}{1 + e^{2x}} \, dx \]

Here, we might try \( u(x) = 1 + e^{2x} \). However, there’s no \( e^{2x} \) available to be the derivative. However, there is an \( e^x \) available, so try

\[ u(x) = e^x \]
\[ du = e^x \, dx \]

\[ \int \frac{e^x \, dx}{1 + e^{2x}} = \int \frac{du}{1 + u^2} = \arctan u + C \]

\[ = \arctan(e^x) + C \]
\[
\int x\sqrt{2-x} \, dx
\]

Here, take \( u(x) \) to be what’s under the radical:
\[ u(x) = 2 - x \quad \text{and} \quad du = - \, dx \]

To do the substitution, we also need to write \( x \) in terms of \( u \):
\[ x = 2 - u \]

\[
\int x\sqrt{2-x} \, dx = \int (2-u)\sqrt{u} \, (-du) = \int (-2u^{1/2} + u^{3/2}) \, du
\]

\[-2(2/3)u^{3/2} + (2/5)u^{5/2} + C
\]

\[= - (4/3)(2-x)^{3/2} + (2/5)(2-x)^{5/2} + C \]