Math 221  
Week 11 part 3

Net Change Theorem  
and some word problems

Please take a moment to just breathe.

Recall: Fundamental Theorem of Calculus (part 2)

If $f$ is continuous on $[a, b]$, then

$$\int_{a}^{b} f(t) \, dt = F(b) - F(a)$$

where $F$ is any antiderivative of $f$.

You may see this notation:

$$\int_{a}^{b} f(t) \, dt = F(x) \bigg|_{x=a}^{x=b} = F(b) - F(a)$$
We restate the FTC slightly to obtain the “Net Change Theorem”:

If $F'(t)$ is the derivative of $F(t)$, then

$$F(b) = F(a) + \int_a^b F'(t)dt$$

For instance, since velocity $v(t)$ is the derivative of the position $p(t)$, we have

$$p(b) = p(a) + \int_a^b v(t)dt$$

Example:
To find your position at time $t = 20$,
figure where you were at time $t = 0$,
and then compute the net distance you traveled:

$$p(20) = p(0) + \int_0^{20} v(t) dt$$

Example. Suppose you jump out of airplane, and your velocity is given by $v(t) = 30(1 - e^{-t})$ feet/sec. How far do you fall in the first 5 seconds?

$$p(5) - p(0) = \int_0^5 v(t) dt$$

$$= \int_0^5 30(1 - e^{-t}) dt$$

$$= 30(x + e^{-x}) \bigg|_0^5$$

$$= 30(5 + e^{-5}) - 30(0 + e^{0})$$

$$= 150 + 30e^{-5} - 30 \approx 120.2$$
Example. Suppose water flows into and out of a tank, and the NET rate of change of the volume in the tank is $V'(t) = 20(t^2 - 1)$ gallons per minute. If it has 300 gallons at time $t = 0$, how much does it have at time $t = 3$ min?

\[
V(3) = V(0) + \int_0^3 V'(t) \, dt
\]

\[
= 300 + 20 \int_0^3 (t^2 - 1) \, dt
\]

\[
= 300 + 20 \left( \frac{t^3}{3} - t \right) \bigg|_0^3
\]

\[
= 300 + 20 \left[ \left( \frac{3^3}{3} - 3 \right) - \left( \frac{0^3}{3} - 0 \right) \right] = 420 \text{ gals}
\]

Example. Suppose the linear density of a 5 meter metal rod is given by $\rho(x) = 4 + x$ kg/m, where $x$ is the distance in meters from the end of the rod. Find the total mass of the rod.

Think of $\rho$ as the rate of change of mass with respect to the length of the rod. The small bit of rod between $x$ and $x + \Delta x$ has mass approximately equal to $\rho(x) \Delta x$.

If we add up all the bits of mass, we get

\[
\sum_{i=1}^{n} \rho(x_i) \Delta x.
\]

Example. Net Distance vs Total Distance traveled.

Suppose your velocity is $v(t) = 2t - 4$, and you travel from $t = 0$ to $t = 5$.

The net change in your position would be

\[
p(5) - p(0) = \int_0^5 v(t) \, dt
\]

\[
= \int_0^5 (2t - 4) \, dt = (t^2 - 4t) \bigg|_0^5
\]

\[
= 25 - 20 - (0 - 0) = 5
\]
Example. Net Distance vs Total Distance traveled.

\[ v(t) = 2t - 4 \] from \( t = 0 \) to \( t = 5 \).

However, to find the total distance travelled, we have to know when we were going forwards and when we were going backwards:

We break the time interval into two separate intervals:
- on \([0,2]\) we are going backwards
- on \([2,5]\) we are going forwards

To compute the total distance traveled, we compute

\[
-\int_{0}^{2} (2t - 4) \, dt + \int_{2}^{5} (2t - 4) \, dt
\]

\[
= -(r^2 - 4t) \Big|_{0}^{2} + (r^2 - 4t) \Big|_{2}^{5} = 4 + 9 = 13
\]