In this section we prove the “first” part of the Fundamental Theorem of Calculus.

Recall the definition:

The **definite integral** of \( f(x) \) from \( x = a \) to \( x = b \) is

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

if this limit exists.

We can think of the integral as the signed area under the curve \( y = f(x) \) between \( x = a \) and \( x = b \).
Fundamental Theorem of Calculus, Part 1:

If $f$ is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t) \, dt$$

is an antiderivative of $f(x)$.

Note that $F(x)$ is a function of $x$, not a function of $t$.

Example.
Suppose that $f(t)$ is the velocity of a plane at time $t$,

and $F(x) = \int_0^x f(t) \, dt$

Then

$$F(10) = \int_0^{10} f(t) \, dt \approx 33$$

which is the area under the curve from $t = 0$ to $t = 10$,
or the distance traveled in that time.

Example.
Suppose that $f(t)$ is the velocity of our plane at time $t$,

and $F(x) = \int_0^x f(t) \, dt$

Then

$$F(20) = \int_0^{20} f(t) \, dt \approx 65$$

which is the area under the curve from $t = 0$ to $t = 20$,
or the distance traveled in that time.

For an arbitrary $x$,

$F(x) = \int_0^x f(t) \, dt$

is the signed area under the graph of $f(t)$ from $t = 0$ to $t = x$,

or the net distance traveled from time $t = 0$ to time $t = x$. 
Now we show that the derivative of the function $F(x)$ is $f(x)$.

We know that

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h}$$

Finally, we use the Squeeze Theorem to compute the limit.

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h}$$

Let $m_h$ be the minimum of $f(x)$ on the interval $[x, x + h]$. Let $M_h$ be the maximum of $f(x)$ on the interval $[x, x + h]$.

Then

$$m_h h \leq \int_{x}^{x+h} f(x) \, dx \leq M_h h$$

so

$$m_h \leq \frac{1}{h} \int_{x}^{x+h} f(x) \, dx \leq M_h$$
Since \( m_h \leq \frac{1}{h} \int_x^{x+h} f(x) \, dx \leq M_h \)

and both \( \lim_{h \to 0} m_h = f(x) \) and \( \lim_{h \to 0} M_h = f(x) \),

\[ F'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(x) \, dx = f(x) \]

and we have proved the first part of the Fundamental Theorem of Calculus.

Examples.
Find the derivative of \( F(x) = \int_0^x t^3 \, dt \)

Find the derivative of \( F(x) = \int_0^x \ln(t^3 + 2 + \cos t) \, dt \)

Try these yourself.

Fundamental Theorem of Calculus, Part 1:
If \( f \) is continuous on \([a, b]\), then the function

\[ F(x) = \int_a^x f(t) \, dt \]

is an antiderivative of \( f(x) \).

Note that we have shown that the integral and the derivative “undo” each other: if \( F \) is defined as above, then

\[ \frac{d}{dx} F(x) = f(x) \]

The derivative of \( F(x) = \int_0^x t^3 \, dt \) is \( F'(x) = x^3 \)

The derivative of \( F(x) = \int_0^x \ln(t^3 + 2 + \cos t) \, dt \) is \( F'(x) = \ln(x^3 + 2 + \cos x) \)

The derivative of \( F(x) = \int_0^x \text{uglyfunction}(t) \, dt \) is \( F'(x) = \text{uglyfunction}(x) \)
Examples.

Find the derivative of \( F(x) = \int_x^0 t^3 \, dt \)

Find the derivative of \( G(x) = \int_0^{x^2} \cos t \, dt \)

There are two ways to do this:

Method 1: Think of this as a composite function:

\[ G(x) = F(x^2), \]

where \( F(x) = \int_0^x \cos t \, dt. \)

Since \( F'(x) = \cos x, \)

we have \( G'(x) = [F(x^2)]' = F'(x^2)(2x) = \cos(x^2)(2x) \)

Method 2: Use the 2nd part of the FTC:

\( G(x) = \int_0^{x^2} \cos t \, dt = \sin(x^2) - \sin(0) \)

so \( G'(x) = [\sin(x^2) - 0]' = \cos(x^2)(2x) \)

In general, if

\( G(x) = \int_a^{u(x)} f(t) \, dt, \quad \text{then} \quad G'(x) = f(u(x))u'(x) \)