Approximating the area under a curve

We show how to chop up the area under a curve into rectangles to approximate it.

Suppose we want to measure the area between the curve $f(x) = x^2$ and the $x$-axis, and between $x = 0$ and $x = 1$. 
To approximate this area, we'll chop it up into vertical rectangles and measure the area of each rectangle.

Here, for example, here we're chopping into five rectangles.

Rectangle 1
width = 0.2, height = f(0.2)

Rectangle 2
width = 0.2, height = f(0.4)

Rectangle 3
width = 0.2, height = f(0.6)

Rectangle 4
width = 0.2, height = f(0.8)

Rectangle 5
width = 0.2, height = f(1.0)

\[ f(x) = x^2 \]

Let's try the computation again, but this time with 10 rectangles:

area of R1 = 0.1 \cdot f(0.1)
area of R2 = 0.1 \cdot f(0.2)
area of R3 = 0.1 \cdot f(0.3)
area of R4 = 0.1 \cdot f(0.4)
area of R5 = 0.1 \cdot f(0.5)
area of R6 = 0.1 \cdot f(0.6)
area of R7 = 0.1 \cdot f(0.7)
area of R8 = 0.1 \cdot f(0.8)
area of R9 = 0.1 \cdot f(0.9)
area of R10 = 0.1 \cdot f(1.0)

Total:

\[ 0.1 \cdot (0.01 + 0.04 + 0.09 + 0.16 + 0.25 + 0.36 + 0.49 + 0.64 + 0.81 + 1.0) = 0.1 \cdot (3.85) = 0.385 \]

Total area: .008 + .032 + .072 + .128 + .200 = 0.440
Intuitively, the more rectangles we use, the better the approximation will be.

If we approximate the area under \( y = f(x) \) between \( x = a \) and \( x = b \), using \( n \) rectangles, then

the \( i \)th rectangle will have

\[
\begin{align*}
\text{width} & = \frac{b - a}{n} \\
\text{height} & = f(a + i \left( \frac{b - a}{n} \right))
\end{align*}
\]

Notation

To deal with these sums, we need “summation notation.”

\[
\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_{n-1} + a_n
\]

“Sum from \( i = 1 \) to \( n \) of \( a_i \)”

start the sum at \( a_1 \)

end the sum at \( a_n \)

You can start and end the sum at any pair of integers.

For instance: \[
\sum_{i=0}^{3} a_i = a_0 + a_1 + a_2 + a_3
\]

Examples

\[
\begin{align*}
\sum_{i=1}^{n} i & = 1 + 2 + 3 + \ldots + n \\
\sum_{i=1}^{n} i^2 & = 1 + 4 + 9 + \ldots + n^2 \\
\sum_{i=0}^{n} 2^{-i} & = 1 + 1/2 + 1/4 + 1/8 + \ldots + 1/2^n
\end{align*}
\]

Our first approximation for the area with 5 rectangles was

\[
\text{Area} \approx 0.2f(0.2) + 0.2f(0.4) + 0.2f(0.6) + 0.2f(0.8) + 0.2f(1.0)
\]

This can be written with summation notation as

\[
\text{Area} \approx \sum_{i=1}^{5} (0.2)f(0.2i)
\]

For our approximation with 10 rectangles, we get

\[
\text{Area} \approx \sum_{i=1}^{10} (0.1)f(0.1i)
\]
To approximate the area under \( y = f(x) \) from \( x = a \) to \( x = b \) using \( n \) rectangles:

The width of each rectangle is

\[ \Delta x = \frac{b - a}{n} \]

The height of the \( i \)-th rectangle is

\[ f(a + i\Delta x) \]

So the total area of the rectangles is

\[ \text{Area} \approx \sum_{i=1}^{n} \Delta x \cdot f(a + i\Delta x) \]

Example. Set up (but do not evaluate) the sum to approximate the area under \( y = \ln(x) \) between \( x = 2 \) and \( x = 5 \) using 10 (or more generally, \( n \)) rectangles.

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Please try this yourself. (Pause the video)

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Note: we can also measure the heights of the rectangles at the left endpoints instead of the right endpoints.
Or, we can measure heights at the midpoints, which usually gives a more accurate approximation since the errors cancel somewhat.

In fact, we can measure the height at an arbitrary point in each subinterval. These points are often written as $x^*_i$.

Here are the general formulas. Let $\Delta x = \frac{b - a}{n}$. Then

Right endpoints: $\text{Area} \approx \sum_{i=1}^{n} \Delta x \, f(a + i\Delta x)$

Left endpoints: $\text{Area} \approx \sum_{i=1}^{n} \Delta x \, f(a + (i - 1)\Delta x)$

Midpoints: $\text{Area} \approx \sum_{i=1}^{n} \Delta x \, f(a + (i - 1/2)\Delta x)$

Finally, if we evaluate the height at arbitrary points $x^*_i$, then we get the formula

$$\text{Area} \approx \sum_{i=1}^{n} \Delta x \, f(x^*_i)$$

In the next section, we'll take the limit as $n$ goes to infinity.