Math 221
Week 10 part 1
Antiderivatives
Please take a moment to just breathe.
We define antiderivatives and show how to compute a few.
Antiderivatives.

If $F(x) = x^3$, then $F'(x) = 3x^2$.

If $G'(x) = 3x^2$, then what was $G(x)$?
If $G'(x) = 3x^2$, then what was $G(x)$?

$G(x)$ could be $x^3$, or $x^3 + 1$, or $x^3 + 2$, or $x^3 + C$ for any constant $C$.

All of these functions have the same derivative!
Definition.

A function \( F(x) \) is an **antiderivative** of the function \( f(x) \) on an interval \( I \) if \( F'(x) = f(x) \) for all \( x \) in \( I \).

Theorem.

Any two antiderivatives of \( f \) differ only by a constant.

Example:

Any antiderivative of \( g(x) = 3x^2 \) must have the form \( G(x) = x^3 + C \), for some constant \( C \).
Example: Find the general form of the antiderivative of these functions (try this yourself!):

\[ f(x) = x^{1/3} \]

\[ g(x) = e^x + \frac{1}{x} + \cos x \]

\[ h(x) = \frac{2}{1 + x^2} \]
**Example:** Find the general form of the antiderivative of these functions:

\[
\begin{align*}
f(x) &= x^{1/3} \\
F(x) &= \frac{3}{4}x^{4/3} + C \\
g(x) &= e^x + \frac{1}{x} + \cos x \\
G(x) &= e^x + \ln |x| + \sin x + C \\
h(x) &= \frac{2}{1 + x^2} \\
H(x) &= 2 \arctan x + C
\end{align*}
\]

For \( x < 0, \quad \frac{d}{dx} \left( \ln(-x) \right) = \frac{-1}{-x} = \frac{1}{x} \)
What if we are given more information?

**Example:** Find $F$ so that $F'(x) = x^{1/3}$ and $F(1) = 2$.

We know $F(x) = \frac{3}{4}x^{4/3} + C$, so we have to find $C$.

$$F(1) = 2$$

$$\frac{3}{4}1^{4/3} + C = 2$$

$$C = \frac{5}{4}$$
Example.

Find $H(x)$ if $H'(x) = \frac{2}{1 + x^2}$ and $H(1) = 0$.

Try this one yourself (pause the video).
Example. Find $H(x)$ if $H'(x) = \frac{2}{1 + x^2}$ and $H(1) = 0$.

We know $H(x) = 2 \arctan x + C$, so we have to find $C$.

\[
H(1) = 0
\]

\[
2 \arctan(1) + C = 0
\]

\[
C = -\frac{\pi}{2}
\]

\[
H(x) = 2 \arctan x - \frac{\pi}{2}
\]
The same procedure works for higher derivatives, but we have to solve for more constants.

Example:

Find the most general form for \( F \) if \( F''(x) = -\frac{1}{x^2} \).

We have

\[
F'(x) = \frac{1}{x} + C
\]

\[
F(x) = \ln |x| + Cx + D
\]
Example.

Find $F$ if $F''(x) = -\frac{1}{x^2}$, and $F(1) = 0$, and $F'(1) = 3$.

We know $F'(x) = \frac{1}{x} + C$ and $F(x) = \ln |x| + Cx + D$,

$F'(1) = 3$, so $\frac{1}{1} + C = 3$, so $C = 2$.

$F(0) = 1$, so $\ln(1) + C + D = 0$, so $D = -2$

So $F(x) = \ln |x| + 2x - 2$. 