Math 221
Week 9 part 1

Optimization
word problems
Please take a moment to just breathe.
We give a step by step method for solving optimization word problems.
Recall

A **critical point** of $f$ is a number $c$ in the domain of $f$ where $f'(c) = 0$ or $f''(c)$ does not exist.

A critical point $c$ gives a local **minimum** if $f''(c) > 0$ or if $f$ is decreasing to the left of $c$ and increasing to the right of $c$.

A critical point $c$ gives a local **maximum** if $f''(c) < 0$ or if $f$ is increasing to the left of $c$ and decreasing to the right of $c$.

For a **closed interval**, the absolute optimum must be at a critical point or an endpoint.
Example.

Suppose you want to build a rectangular llama pen along a river using only 100 m of fencing. You plan to fence off the three sides that are not along the river.

What shape should the pen be to maximize the area of the pen?
General Method for solving optimization problems.

Step 1. Draw a picture and give things names.
Step 2. Which quantity do we want to optimize?

Step 3. What bounds are naturally given by the problem?
Step 4. What constraints are given? What relationships between quantities are given?

Step 5. Write the quantity you want to optimize in terms of just one other variable.

Then we are ready do to some calculus!
Step 6. Take the derivative and find the critical points.

Step 7. Use the first or second derivative test to see which critical point is optimal. Also check endpoints, if relevant.

Step 8. Do a reality check: does this answer make sense given my intuition about the real world?
Suppose you want to build a rectangular llama pen along a river using only 100 m of fencing to fence off the other three sides. What shape should the pen be to maximize the area of the pen?

**Step 1. Draw a picture and give things names.**

**Step 2. Which quantity do we want to optimize?**

\( x \) is the length of the sides perpendicular to the water
\( y \) is the length of fence parallel to the water
\( A \) is the area in the pen.

We want to maximize the area \( A \).
Suppose you want to build a rectangular llama pen along a river using only 100 m of fencing to fence off the other three sides. What shape should the pen be to maximize the area of the pen?

**Step 3. What bounds arise naturally?**

\[0 \leq 2x \leq 100\]
\[0 \leq y \leq 100\]

**Step 4. Write down constraints or relationships:**

\[A = xy, \quad \text{by the definition of area.}\]
\[2x + y = 100 \quad \text{since we only have 100m of fence.}\]
Step 5. Write the variable you want to optimize in terms of just one other variable.
We want to maximize the area $A$.

$$A = xy$$
$$2x + y = 100$$

Solve for $y$ in the constraint equation:

$$y = 100 - 2x$$

Plug in for $y$ in the area equation:

$$A = x(100 - 2x)$$

Now we have $A$ written as a function of just one variable.
Step 6. Find the critical points of the function \( A(x) \).

\[ A(x) = x(100 - 2x) = 100x - 2x^2 \]

\[ \frac{dA}{dx} = 100 - 4x \]

Critical point: 100 - 4x = 0 when \( x = 25 \).

Step 7. Check to see if the critical point is a max or min.

\[ \frac{d^2A}{dx^2} = -4 < 0 \]

Since the function is concave down everywhere, we know that the area \( A \) is maximized at \( x = 25 \).
Suppose you want to build a rectangular llama pen along a river using only 100 m of fencing to fence off the other three sides. What shape should the pen be to maximize the area of the pen?

**Step 8. Reality check.**

We found the maximum area should occur with $x = 25$. In this case, $y = 100 - 2x = 50$.

This answer seems reasonable!
Example. Suppose you want to make an open (no top) box with a square bottom from two different materials: the sides will be made of stuff costing 1$ per square foot, and the bottom will be made with stuff costing 4$ per square foot.

Find the dimensions of the box of greatest volume that can be made for 100$.

Please try this one yourself. (Pause the video!)
Suppose you want to make an open (no top) box with a square bottom from two different materials: the sides will be made of stuff costing 1$ per square foot, and the bottom will be made with stuff costing 4$ per square foot.

Find the dimensions of the box of greatest volume that can be made for 100$.

**Step 1. Draw a picture and give things names.**

**Step 2. Which quantity do we want to optimize?**

- $x$ is the length of the bottom edge of the box
- $y$ is the height of the box
- $V$ is the volume inside the box

We want to maximize $V$. 
Step 3. Natural bounds
$x \geq 0$, and $y \geq 0$. There are other bounds, but we’ll start with these.

Step 4. Write down constraints or relationships:

- surface area of one side: $xy$
- cost of the four sides: $4xy(1)$

- surface area on the bottom: $x^2$
- cost of the bottom: $x^2(4)$

- total cost of the materials: $C = 4xy + 4x^2 = 100$
Step 5. Write $V$ in terms of just one variable.

$$4xy + 4x^2 = 100$$

It’s easier to solve for $y$:

$$y = \frac{100 - 4x^2}{4x}$$

We know that $V = x^2y$, so

$$V = x^2y = x^2 \frac{100 - 4x^2}{4x} = 25x - x^3$$
Step 6. Find the critical points.

\[ V(x) = 25x - x^3 \]

\[ \frac{dV}{dx} = 25 - 3x^2 \]

\[ 25 - 3x^2 = 0 \text{ when } x = \frac{5}{\sqrt{3}} \]

Step 7. Check the concavity at the critical points.

\[ \frac{d^2V}{dx^2} = -6x, \text{ which is negative at } x = \frac{5}{\sqrt{3}}. \]

So we have found \( x \) that produces the maximum volume.
If $x = \frac{5}{\sqrt{3}}$, then $y = \frac{25 - 25/3}{5/\sqrt{3}} = \frac{10}{\sqrt{3}}$

These dimensions seem reasonable for a box.