Math 221
Week 8 part 3

The
Second Derivative Test

Please take a moment to just breathe.

Definition. A function $f$ is **concave up** on an interval $I$ if the graph of $y = f(x)$ lies above its tangent lines for all $x$ in $I$.

Both of these functions are concave up on $(-\infty, \infty)$.

$$f(x) = x^2$$

$$f(x) = e^x$$
Definition. A function $f$ is **concave up** on an interval $I$ if the graph of $y = f(x)$ lies above its tangent lines for all $x$ in $I$.

A function $f$ is **concave down** on an interval $I$ if the graph of $y = f(x)$ lies below all of its tangent lines for all $x$ in $I$.

“Concave up, like a cup”            “Concave down, like a frown”

Alternative definition:

$f$ is concave up where the slope of its tangent line is increasing.

$f$ is concave down where the slope of its tangent line is decreasing.

Connection to the second derivative:

If $f''(x) > 0$ for all $x$ in an interval $I$, then $f$ is concave up on $I$.

If $f''(x) < 0$ for all $x$ in an interval $I$, then $f$ is concave down on $I$.

Recall the definition:

A **critical point** of $f$ is a number $c$ in the domain of $f$ such that either

$$f'(c) = 0$$

or

$$f'(c)$$

does not exist

Local maxima and minima must occur at critical points, by Fermat's theorem.
Second Derivative Test for determining if a critical point is a local max or min:

Suppose \( c \) is a critical point for \( f \).

If \( f''(c) > 0 \), then \( f \) has a local minimum at \( c \).

If \( f''(c) < 0 \), then \( f \) has a local maximum at \( c \).

If \( f''(c) = 0 \), the test has failed, and we need more information to determine the behavior of \( f \) at \( c \).

Example: Find the local max, min of \( f(x) = x^2 \ln x \)

Step 1. Find the critical points.

\[ f'(x) = 2x \ln x + x^2(1/x) = x(2 \ln x + 1) \]

\[ f'(x) = 0 \] when \( x = 0 \), but \( 0 \) is not in the domain of \( f \), so it’s not a critical point.

\[ f'(x) = 0 \] again when
\[ 2 \ln x + 1 = 0 \]
\[ \ln x = -1/2 \]
\[ x = e^{-1/2} \]

So the only critical point of \( f \) is \( e^{-1/2} \).

Step 2. Check the second derivative at each critical point.

\[ f''(x) = 2 \ln x + 2x(1/x) + 1 = 2 \ln x + 3 \]

\[ f''(e^{-1/2}) = 2 \ln e^{-1/2} + 3 = 2(-1/2) + 3 = 2 \]

Since \( f''(e^{-1/2}) > 0 \), the function is concave up, and there is a local minimum at that point.

Here is a sketch of the graph of \( f(x) = x^2 \ln x \).
Example. Find the local maxima and minima of

\[ f(x) = x^3 + 3x^2 + 3x + 1 \]

Please try this yourself. (Pause the video!)

Step 1. Find the critical points.

\[ f'(x) = 3x^2 + 6x + 3 = 3(x + 1)^2 \]
\[ f'(x) = 0 \quad \text{when} \quad x = -1. \]

Step 2. Check the second derivative at each critical point.

\[ f''(x) = 6x + 6 \]
\[ f''(-1) = 0 \]

The second derivative test has FAILED.

Since the second derivative test has failed, we go back to the first derivative test:

The critical point is at \( x = -1 \).

To the left of \(-1\), \( f'(x) > 0 \), so the function is increasing.

To the right of \(-1\), \( f'(x) > 0 \), so the function is increasing.

By the first derivative test, \( x = -1 \) is neither a local maxima nor a local minima.

Inflection points

We say that \( b \) is an \textit{inflection point} for \( f \) if

\[ f''(x) > 0 \quad \text{on one side of } x = b, \quad \text{and} \]
\[ f''(x) < 0 \quad \text{on the other side.} \]

Note: in this case, either \( f''(b) = 0 \) or \( f''(b) \) does not exist.
Example:
Both of these curves have an inflection point at \( x = 0 \):

\[
\begin{align*}
 f(x) &= x^3 \\
 f'(x) &= 3x^2 \\
 f''(x) &= 6x \\
 g(x) &= x^3 - x \\
 g'(x) &= 3x^2 - 1 \\
 g''(x) &= 6x
\end{align*}
\]