Math 221
Week 8 part 2

The First Derivative Test

Please take a moment to just breathe.

We define what it means for a function to be increasing or decreasing.

We state the First derivative test and explain how to use it to check if a critical point is a maximum, minimum, or neither.

Definition
A function \( f \) is increasing on an interval \( I \) if
\[ f(x) \leq f(y) \text{ whenever } x \leq y \text{ in } I. \]

A function \( f \) is decreasing on an interval \( I \) if
\[ f(x) \geq f(y) \text{ whenever } x \leq y \text{ in } I. \]
Facts:
If $f'(x) > 0$ for all $x$ in an interval $I$, then $f$ is increasing on $I$.
If $f'(x) < 0$ for all $x$ in an interval $I$, then $f$ is decreasing on $I$.

Recall the definition:
A critical point of $f$ is a number $c$ in the domain of $f$ such that either
$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist}$$

Local maxima and minima must occur at critical points, by Fermat’s theorem.

First Derivative Test
Suppose $c$ is a critical point for $f$.

If $f$ is increasing to the left of $c$ and decreasing to the right of $c$, then $f$ has a local maximum at $c$.

On the other hand, if $f$ is decreasing to the left of $c$ and increasing to the right of $c$, then $f$ has a local minimum at $c$.

Example: Find the local max, min of $f(x) = x^{10}(x - 1)^9$

Step 1. Find the critical points.
$$f'(x) = 10x^9(x - 1)^9 + 9x^{10}(x - 1)^8$$
$$= x^9(x - 1)^8[10(x - 1) + 9x]$$
$$= x^9(x - 1)^8[19x - 10]$$
$$f'(x) = 0 \text{ when }$$
$$x = 0$$
$$x = 1$$
$$x = 10/19$$
Step 2. Divide the number line into segments between the critical points.

The sign of the derivative can only change at the critical points. We can check the sign of \( f' \) at a “nice” point in each interval.

\[
\begin{array}{cccc}
\text{try} & \text{try} & \text{try} & \text{try} \\
\text{x = 1} & \text{x = 1/4} & \text{x = 3/4} & \text{x = 2} \\
\text{x = 0} & \text{x = 10/19} & \text{x = 1} & \\
\end{array}
\]

Step 3. Determine where \( f \) is increasing and decreasing.

\[
f'(x) = x^9(x - 1)^8[19x - 10]
\]

- \( f'(-1) \) has factors that are \((-)(+)(-)\), so \( f'(-1) > 0 \)
- \( f'(1/4) \) has factors that are \((+)(+)(-)\), so \( f'(1/4) < 0 \)
- \( f'(3/4) \) has factors that are \((+)(+)(+)\), so \( f'(3/4) > 0 \)
- \( f'(2) \) has factors that are \((+)(+)(+)\), so \( f'(2) > 0 \)

Here is the graph of \( f(x) = x^{10}(x - 1)^9 \). Note how far we had to zoom in on the y-axis to see the bump for the minimum.

Example: Find local maxima and minima of

\[
f(x) = x^{2/3}(6 - x)^{1/3}
\]

Please try this one on your own. (Pause the video.)
Example: Find local maxima and minima of
\[ f(x) = x^{2/3}(6 - x)^{1/3} \]

\[ f'(x) = \frac{2}{3} x^{-1/3}(6 - x)^{1/3} - x^{2/3} \left( \frac{1}{3} (6 - x)^{-2/3} \right) \]
\[ = \frac{2(6 - x) - x}{3x^{1/3}(6 - x)^{2/3}} = \frac{12 - 3x}{3x^{1/3}(6 - x)^{2/3}} \]

So \( f \) has critical points at \( x = 0, x = 4, \) and \( x = 6. \)

We need to check the sign of \( f' \) on the intervals \((-\infty, 0), (0, 4), (4, 6), \) and \((6, \infty)\).

Here’s a sketch of the graph of \( f(x) = x^{2/3}(6 - x)^{1/3} \).

You can see the local maximum and minima and where the graph has infinite slope.

We need to check the sign of \( f'' \) on the intervals \((-\infty, 0), (0, 4), (4, 6), \) and \((6, \infty)\).

\[ f''(x) = \frac{12 - 3x}{3x^{1/3}(6 - x)^{2/3}} \]

\( f'(-1) \) has factors \(+\)/(−)/(+), so \( f'(-1) < 0 \)
\( f'(1) \) has factors \(+\)/(+)/(+), so \( f'(1) > 0 \)
\( f'(5) \) has factors \(−\)/(+)/(+), so \( f'(5) < 0 \)
\( f'(7) \) has factors \(−\)/(+)/(+), so \( f'(7) < 0 \)

So \( f \) has a local minimum at \( x = 0 \)
and a local maximum at \( x = 4.\)