Math 221
Week 8 part 2
The First Derivative Test
Please take a moment to just breathe.
We define what it means for a function to be increasing or decreasing.

We state the First derivative test and explain how to use it to check if a critical point is a maximum, minimum, or neither.
Definition
A function $f$ is **increasing** on an interval $I$ if $f(x) \leq f(y)$ whenever $x \leq y$ in $I$.

A function $f$ is **decreasing** on an interval $I$ if $f(x) \geq f(y)$ whenever $x \leq y$ in $I$. 

![Graph showing increasing and decreasing intervals](image)
Facts:
If $f''(x) > 0$ for all $x$ in an interval $I$, then $f$ is **increasing** on $I$.
If $f''(x) < 0$ for all $x$ in an interval $I$, then $f$ is **decreasing** on $I$. 
Recall the definition:

**A critical point** of $f$ is a number $c$ in the domain of $f$ such that either

$$f'(c) = 0$$

or

$$f''(c)$$

does not exist

Local maxima and minima must occur at critical points, by Fermat’s theorem.
First Derivative Test
Suppose $c$ is a critical point for $f$.

If $f$ is increasing to the left of $c$ and decreasing to the right of $c$, then $f$ has a local maximum at $c$.

On the other hand, if $f$ is decreasing to the left of $c$ and increasing to the right of $c$, then $f$ has a local minimum at $c$. 
Example: Find the local max, min of \( f(x) = x^{10}(x - 1)^9 \)

**Step 1. Find the critical points.**

\[
f''(x) = 10x^9(x - 1)^9 + 9x^{10}(x - 1)^8
\]

\[
= x^9(x - 1)^8[10(x - 1) + 9x]
\]

\[
= x^9(x - 1)^8[19x - 10]
\]

\[f''(x) = 0 \text{ when}
\]

\[
x = 0
\]

\[
x = 1
\]

\[
x = 10/19
\]
Step 2. Divide the number line into segments between the critical points.

The sign of the derivative can only change at the critical points. We can check the sign of $f'$ at a “nice” point in each interval.
Step 3. Determine where $f'$ is increasing and decreasing.

$$f'(x) = x^9(x - 1)^8[19x - 10]$$

$f'(-1)$ has factors that are $(-)(+)(-)$, so $f'(-1) > 0$

$f'(1/4)$ has factors that are $(+)(+)(-)$, so $f'(1/4) < 0$

$f'(3/4)$ has factors that are $(+)(+)(+)$, so $f'(3/4) > 0$

$f'(2)$ has factors that are $(+)(+)(+)$, so $f'(2) > 0$

\[\text{max} \quad \text{min} \quad \text{neither}\]

\[x=0 \quad x=\frac{10}{19} \quad x=1\]
Here is the graph of \( f(x) = x^{10}(x - 1)^9 \). Note how far we had to zoom in on the y-axis to see the bump for the minimum.
Example: Find local maxima and minima of

\[ f(x) = x^{2/3}(6 - x)^{1/3} \]

Please try this one on your own. (Pause the video.)
Example: Find local maxima and minima of

\[ f(x) = x^{2/3}(6 - x)^{1/3} \]

\[ f'(x) = \frac{2}{3} x^{-1/3} (6 - x)^{1/3} - x^{2/3} \frac{1}{3} (6 - x)^{-2/3} \]

\[ = \frac{2(6 - x) - x}{3x^{1/3}(6 - x)^{2/3}} = \frac{12 - 3x}{3x^{1/3}(6 - x)^{2/3}} \]

So \( f \) has critical points at \( x = 0, x = 4, \) and \( x = 6. \)

We need to check the sign of \( f' \) on the intervals \( (-\infty, 0), (0, 4), (4, 6), \) and \( (6, \infty) \).
We need to check the sign of \( f' \) on the intervals \((-\infty, 0), (0, 4), (4, 6), \) and \((6, \infty)\).

\[
f'(x) = \frac{12 - 3x}{3x^{1/3}(6 - x)^{2/3}}
\]

\( f'(-1) \) has factors \((+)/(−)/(+), \) so \( f'(-1) < 0 \)

\( f'(1) \) has factors \((+)/(+)/(+), \) so \( f'(1) > 0 \)

\( f'(5) \) has factors \((-)/(+)/(+), \) so \( f'(5) < 0 \)

\( f'(7) \) has factors \((-)/(+)/(+), \) so \( f'(7) < 0 \)

So \( f \) has a local minimum at \( x = 0 \)

and a local maximum at \( x = 4 \).
Here’s a sketch of the graph of $f(x) = x^{2/3}(6 - x)^{1/3}$.

You can see the local maximum and minima and where the graph has infinite slope.