Math 221  
Week 8 part 1

Finding maxima and minima on a closed interval

We explain the difference between global maxima and minima and local maxima and minima.

We define critical points.

We discuss how to find the absolute maximum and minimum on a closed interval.

Definition

A function $f$ has a **global maximum** at $c$ if $f(c) \geq f(x)$ for all $x$ in the domain of $f$.

A function $f$ has a **local maximum** at $c$ if $f(c) \geq f(x)$ for all $x$ in an open interval containing $c$.

**Global minima** and **local minima** are defined similarly.
local minimum and absolute minimum

no absolute maximum since function gets larger and larger as \( x \) goes to infinity

Example: \( f(x) = x^2 \) with domain \([-1, 2] \):

At \( x = 2 \), \( f \) has an absolute maximum. This is not a local minimum, however, since 2 is an endpoint.

At \( x = 0 \), \( f \) has an absolute minimum and a local minimum.

Definition:

A critical point of \( f \) is a number \( c \) in the domain of \( f \) such that either
\[
f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist}
\]

Critical points are candidates for local maxima and minima.

To find the absolute max or min of a function \( f \) defined on a closed interval \([a, b]\), we follow the following algorithm.

1) Find all the critical points on \( f \) that lie in the interval.

2) Check the values of \( f \) at all the critical points and both endpoints.

3) The biggest value is the absolute maximum, and the smallest value is the absolute minimum.
Example: Find the absolute maximum of \( f(x) = 4x^{3/5} - x^{8/5} \) on the interval \([-1, 4]\).

**Step 1. Find the critical points.**

\[
f'(x) = \frac{12}{5} x^{-2/5} - \frac{8}{5} x^{3/5} = \frac{x^{-2/5}}{5} [12 - 8x].
\]

so

\[
f'(x) = 0 \text{ when } x = 1.5
\]

\[
f''(x) \text{ is undefined when } x = 0
\]

Our two critical points are \( x = 0 \) and \( x = 1.5 \).

Find the absolute maximum of \( f(x) = 4x^{3/5} - x^{8/5} \) on the interval \([-1, 4]\).

Next we evaluate \( f \) at each of the critical points and both endpoints.

\[
f(-1) = 4(-1)^{3/5} - (-1)^{8/5} = -5
\]

\[
f(0) = 0
\]

\[
f(1.5) = 4(1.5)^{3/5} - (1.5)^{8/5} \approx 3.189
\]

\[
f(4) = 4(4)^{3/5} - (4)^{8/5} = 0
\]

The function has a global maximum at \( x = 1.5 \).

What guarantees do we have that our algorithm works?

Does there have to be an absolute max/min at all?

1. **Extreme Value Theorem**

If \( f \) is continuous on the closed interval \([a, b]\), then it attains an absolute maximum and minimum on that interval.

So we know the absolute max/min must exist.

The absolute maximum of \( f(x) = 4x^{3/5} - x^{8/5} \) on the interval \([-1, 4]\) is at \( x = 1.5 \).

It also has a vertical tangent line at \( x = 0 \).
Now we know that the absolute max/min must exist. Where can they be?

2. Fermat's Theorem
If $f$ has a local maximum or minimum at $x = c$, then either $f'(c) = 0$ or $f'(c)$ does not exist.

(In other words, local maxima and minima have to be at critical points.)

3. Fact: if the absolute max (min) must be either at an endpoint or at a local max or min.

(If it’s at any other point, then we could increase (decrease) $f$ by increasing or decreasing $x$ slightly.

So we only need to check the critical points and endpoints.

Fermat is more famous for his “Last Theorem”:
In 1637, the mathematician Pierre de Fermat believed he had proved the following theorem:

For $n > 2$, there are no integers $a, b,$ and $c$ such that

$$a^n + b^n = c^n$$

However, he did not write down the proof. Instead, he left this message in one of his notebooks:

“I have a truly marvelous demonstration of this proposition which this margin is too narrow to contain.”

Mathematicians obsessed over this theorem for the next 350 years. It was finally proven in 1994 by Andrew Wiles.

Why does interval need to be closed?
Consider $f(x) = x^2$ with domain the open interval $(-1, 2)$:

Here, $f$ gets closer and closer to 4 as $x$ approaches 2, but it never gets to 4. There’s no absolute maximum, just values closer and closer to 4.

See https://simonsingh.net/books/fermats-last-theorem/the-whole-story/ for more of the crazy history of this problem.