Math 221  
Week 7 part 1  

Linear approximation

We use a function’s tangent line as an approximation.

We state linear approximations for $(1 + x)^r$, $e^x$, and $\sin x$ for $x$ near 0, and show how to combine them.

Please take a moment to just breathe.

Very Useful Fact:

If $f$ is differentiable at $x = a$, then

$$f(a + h) = f(a) + f'(a)h + E(h)h,$$

where $E(h)$ is a function with limit 0 at 0.

We can rewrite this equation as

$$f(a + h) \approx f(a) + f'(a)h$$

where the error of the approximation has size $E(h)h$. 
**Very Useful Fact, restated with “x-a” for “h”**

\[ f(x) \approx f(a) + f'(a)(x - a) \]

where the error of the approximation has size

\[ E(x - a)(x - a). \]

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**Linear approximation.**

Idea: the tangent line to a graph is a good approximation near the point of tangency.

Suppose \( f(x) \) is differentiable at \( x = a \). Then the **linearization** of \( f(x) \) is a new function

\[ L(x) = f(a) + f'(a)(x - a) \]

By the Very Useful Fact,

\[ L(x) \approx f(x) \text{ for } x \text{ near } a. \]

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Example: Let \( f(x) = \sqrt{x} \).

The linearization of \( f(x) \) near \( a = 9 \) is

\[ L(x) = f(a) + f'(a)(x - a) = \sqrt{9} + \frac{1}{2\sqrt{9}}(x - 9) \]

or

\[ L(x) = 3 + \frac{1}{6}(x - 9) \]

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Example: Find \( \sqrt{9.03} \)

Let \( f(x) = \sqrt{x} \) and let \( a = 9 \) again.

The linearization of \( f(x) \) is

\[ L(x) = 3 + \frac{1}{6}(x - 9) \]

We evaluate the linearization at our point:

\[ L(9.03) = 3 + \frac{1}{6}(9.03 - 9) = 3.005 \]

Compare with the actual square root:

\[ \sqrt{9.03} = 3.00499584026 \]
More generally, we can show that
\[(1 + h)^r \approx 1 + hr\]
for any exponent \(r\) and for \(h\) near 0.

Proof:
Let \(f(x) = (1 + x)^r\).
\[f'(x) = r(1 + x)^{r-1}\]
\[f'(0) = r(1 + 0)^{r-1} = r\]

By the Very Useful Fact,
\[f(h) \approx f(0) + f'(0)h = 1 + rh\]

Example: Find \(\sqrt[3]{9.03}\)
Alternative solution using \((1 + h)^r \approx 1 + hr\)

Step 1: rewrite the problem so it has the right form:
\[\sqrt[3]{9.03} = \sqrt[3]{9 + \frac{3}{100}} = \sqrt[3]{9 \left(1 + \frac{1}{300}\right)} = 3 \left(1 + \frac{1}{300}\right)^{1/3}\]

Step 2: use the approximation:
\[3 \left(1 + \frac{1}{300}\right)^{1/3} \approx 3 \left(1 + \frac{1}{2 \cdot 300}\right) = 3 + \frac{1}{200} = 3.005\]

Example: Approximate the cube root of 1012.

Please try this problem yourself. (Pause the video!)

Example: Approximate the cube root of 1012.
\[\sqrt[3]{1012} = 10 \sqrt[3]{1 + .012}\]
\[\approx 10 \left(1 + \frac{.012}{3}\right) = 10(1.004) = 10.04\]
Example: Approximate $\cos(1)$ using linear approximation.

Let $f(x) = \cos x$, and let $a = \pi/3$, which is fairly close to 1.

$L(x) = f(a) + f'(a)(x - a)$

$= \cos(\pi/3) - \sin(\pi/3)(x - \pi/3)$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \pi/3)$

$L(1) = \frac{1}{2} - \frac{\sqrt{3}}{2}(1 - \pi/3) = .5403$

Here are three very useful approximations. You should use the Very Useful Fact to convince yourself that they are true.

1. $(1 + x)^r \approx 1 + xr$
   when $x$ is close to 0.

2. $e^x \approx 1 + x$
   when $x$ is close to 0.

3. $\sin x \approx x$
   when $x$ is close to 0.

You can also combine these approximations:

For example, to approximate $f(x) = (4 + \sin x)^{5/2}$ for $x$ near 0:

$(4 + \sin x)^{5/2} \approx (4 + x)^{5/2} = 32(1 + x/4)^{5/2} \approx 32(1 + \frac{5x}{8})$

In Calculus II, you’ll learn the following fact:

If $f$ has first and second derivatives at $x = a$, then

$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2!} f''(a)(x - a)^2$

Here, instead of using a line to approximate $f$, we are using a parabola. This parabola has exactly the same value, first derivative, and second derivative as $f$ does at $x = a$.

This parabola is the “best” parabola for approximating $f$ near $x = a$. 
If $f$ has first, second, and third derivatives at $x = a$, then

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2!} f''(a)(x - a)^2 + \frac{1}{3!} f'''(a)(x - a)^3$$

Here, we use the cubic function that has the same value and the same first, second, and third derivates at $a$. 