Some applications

Example. Suppose the position of an object is given by

\[ p(t) = t^3 - 6t^2 + 9t. \]

Find the total distance travelled in the 5 seconds between \( t = 0 \) and \( t = 5 \).

The object is moving forwards when the derivative is positive.

The object is moving backwards when the derivative is negative.

The total distance is the distance forwards plus the distance backwards.
When does the object change direction?
\[ p(t) = t^3 - 6t^2 + 9t \]
\[ p'(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) \]
\[ p'(t) = 0 \text{ when } 3(t - 1)(t - 3) = 0, \text{ or at } t = 1 \text{ and } t = 3. \]

When is the object moving forwards/backwards?
\[ p'(t) > 0 \text{ on the interval } (0,1): \text{ going forwards for 4} \]
\[ p'(t) < 0 \text{ on the interval } (1,3): \text{ going backwards for 4} \]
\[ p'(t) > 0 \text{ on the interval } (3,5): \text{ going forwards for 20} \]

What is the total distance?
\[ 4 + 4 + 20 = 28. \]

Example: Suppose that to make \( n \) Widgets, you must spend \[ c(n) = 10,000 + 5n + 0.01n^2 \] dollars. This is the “cost function.”

Find the marginal cost at \( n = 500 \) and use it to estimate the cost of making the 501st Widget.

Note: the cost function is not actually defined except at integers, since it does not make sense to make 1/3 of a widget. We pretend that the function is continuous so we can use calculus.

The marginal cost \( c'(n) \) is the rate of change of the cost with respect to the number of objects.
\[ c(n) = 10,000 + 5n + 0.01n^2 \]
\[ c'(n) = 5 + 0.02n \]
\[ c'(500) = 5 + 0.02(500) = 15 \]

The cost of making the 501st widget is the difference between \( c(500) \) and \( c(500 + 1) \). We could work those numbers out, but instead, we use the Very Useful Fact:
\[ c(500 + 1) - c(500) \approx c'(500)(1) \]

It costs about 15 dollars to make the 501st object.
Example: A tank holds 500 gallons of water, which drains from the bottom.

Torricelli’s law says that the volume of the water remaining after $t$ seconds is

$$V(t) = 5000 \left(1 - \frac{t}{40}\right)^2$$

for $t$ in $[0, 40]$. When is the water flowing out the fastest?

Using common sense, we expect the water to be flowing fastest at the very beginning. Let’s see if this works out mathematically.

$$V(t) = 5000(1 - t/40)^2$$

We use the chain rule to compute the derivative:

$$V'(t) = 5000 \cdot 2(1 - t/40) \cdot (1/40)$$

This function is a straight line, so its value on the interval $[0, 40]$ is largest at one end or the other. We check the endpoints:

$$V'(0) = 5000 \cdot 2 \cdot (1/40) = 250 \text{ gallons per second}$$
$$V'(40) = 5000 \cdot 2(1 - 40/40) \cdot (1/40) = 0 \text{ gallons per second}$$

As expected, $V'(t)$ is largest when $t = 0$. 