Math 221
Week 5 part 1

Implicit differentiation

Please take a moment to just breathe.

Equations of curves
Consider the unit circle $x^2 + y^2 = 1$.
The graph is the set of all points $(x, y)$ that satisfy the equation.

We discuss implicit differentiation and logarithmic differentiation.
However, there’s no formula of the form \( y = f(x) \) that describes the entire curve. Instead, we need two functions:

- \( y = \sqrt{1 - x^2} \) for the top half circle
- \( y = -\sqrt{1 - x^2} \) for the bottom half circle

**Method of Implicit differentiation (for implicit curves in variables \( x \) and \( y \))**

Step 1. Take the derivative with respect to \( x \) on both sides of the equation.
Step 2. Solve for \( \frac{dy}{dx} \).

Find the derivative \( \frac{dy}{dx} \) for the curve \( x^2 + y^2 = 1 \).

**Step 1. Take the derivative with respect to \( x \) on both sides:**

\[
\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[1] \\
2x + \frac{d}{dx}[y^2] = 0 \\
2x + 2y \frac{dy}{dx} = 0 \\
2x + 2y \frac{dy}{dx} = 0
\]

Think of \( y \) as a function of \( x \) and use chain rule:

\[
\frac{d}{dx}[y^2] = 2y \frac{dy}{dx}
\]

**Step 2. Solve for \( \frac{dy}{dx} \).**

\[
2x + 2y \frac{dy}{dx} = 0 \\
2y \frac{dy}{dx} = -2x \\
\frac{dy}{dx} = \frac{-x}{y}
\]
Example. Find the tangent line to the curve $x^2 + y^2 = 1$ at the point $(1/2, \sqrt{3}/2)$.

We calculated that \( \frac{dy}{dx} = -\frac{x}{y} \) at the point \((x, y)\).

The slope of the tangent line at the point \((1/2, \sqrt{3}/2)\) is \( \frac{dy}{dx} = -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}} \).

The equation of the tangent line is \( y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}(x - \frac{1}{2}) \).

Example. Folium of Descartes

In France, in the 17th century, mathematicians would exchange challenges for fun.

Fermat claimed: I can find a tangent line to anything!

Descartes responded: Oh yeah? Try this: \( x^3 + y^3 = 3xy \)

Folium of Descartes

Use the method of implicit differentiation to find \( \frac{dy}{dx} \) for \( x^3 + y^3 = 3xy \)

(Please pause the video and try it yourself.)

Folium of Descartes

Use the method of implicit differentiation to find \( \frac{dy}{dx} \).

Step 1. Take derivatives on both sides.

\[
\frac{d}{dx} [x^3 + y^3] = 3x^2 \frac{dy}{dx}
\]

Product and Chain rules:

\[
\frac{d}{dx} [xy] = (1)y + x \frac{dy}{dx}
\]

\[
3x^2 + 3y^2 \frac{dy}{dx} = 3(1)y + 3x \frac{dy}{dx}
\]
**Folium of Descartes**

Step 2. Solve for $\frac{dy}{dx}$:

$$3x^2 + 3y^2 \frac{dy}{dx} = 3(1)y + 3x \frac{dy}{dx}$$

$$x^2 + y^2 \frac{dy}{dx} = y + x \frac{dy}{dx}$$

$$y^2 \frac{dy}{dx} - x \frac{dy}{dx} = y - x^2 \quad \text{so} \quad \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

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**Find the mistake:**

Use implicit differentiation on

$$2y + \cos x + x^2y + \sin(4y) = 1$$

Step 1: We take the derivative with respect to $x$:

INCORRECT answer:

$$2 - \sin x + 2x \frac{dy}{dx} + 4 \cos(4y) \frac{dy}{dx} = 1$$

Where are the three mistakes?
(Please pause the video and look for them.)

CORRECT answer:

$$2 \frac{dy}{dx} - \sin x + 2xy + x^2 \frac{dy}{dx} + 4 \cos(4y) \frac{dy}{dx} = 0$$

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Use implicit differentiation on

$$2y + \cos x + x^2y + \sin(4y) = 1$$

Step 1: We take the derivative with respect to $x$:

INCORRECT answer:

$$2 - \sin x + 2x \frac{dy}{dx} + 4 \cos(4y) \frac{dy}{dx} = 1$$

- forgot to use chain rule
- incorrect use of product rule
- derivative of a constant is 0

CORRECT answer:

$$2 \frac{dy}{dx} - \sin x + 2xy + x^2 \frac{dy}{dx} + 4 \cos(4y) \frac{dy}{dx} = 0$$