Math 221
Week 5 part 1

Implicit differentiation
Please take a moment to just breathe.
We discuss implicit differentiation and logarithmic differentiation.
Equations of curves
Consider the unit circle \( x^2 + y^2 = 1 \).
The graph is the set of all points \((x, y)\) that satisfy the equation.
However, there’s no formula of the form $y = f(x)$ that describes the entire curve. Instead, we need two functions:

- $y = \sqrt{1 - x^2}$ for the top half circle
- $y = -\sqrt{1 - x^2}$ for the bottom half circle
Method of Implicit differentiation (for implicit curves in variables $x$ and $y$)

Step 1. Take the derivative with respect to $x$ on both sides of the equation.

Step 2. Solve for $\frac{dy}{dx}$. 
Find the derivative $\frac{dy}{dx}$ for the curve $x^2 + y^2 = 1$.

**Step 1.** Take the derivative with respect to $x$ on both sides:

$$x^2 + y^2 = 1$$

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[1]$$

$$2x + \frac{d}{dx}[y^2] = 0$$

Think of $y$ as a function of $x$ and use chain rule:

$$\frac{d}{dx}[y^2] = 2y \frac{dy}{dx}$$

$$2x + 2y \frac{dy}{dx} = 0$$
Step 2. Solve for $\frac{dy}{dx}$.

\[2x + 2y \frac{dy}{dx} = 0\]

\[2y \frac{dy}{dx} = -2x\]

\[
\frac{dy}{dx} = -\frac{x}{y}
\]
Example. Find the tangent line to the curve \( x^2 + y^2 = 1 \) at the point \((1/2, \sqrt{3}/2)\).

We calculated that \( \frac{dy}{dx} = -\frac{x}{y} \) at the point \((x, y)\).

The slope of the tangent line at the point \((1/2, \sqrt{3}/2)\)

is \( \frac{dy}{dx} = -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}} \)

The equation of the tangent line is

\[
y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}(x - \frac{1}{2})
\]
Example. **Folium of Descartes**

In France, in the 17th century, mathematicians would exchange challenges for fun.

Fermat claimed: I can find a tangent line to anything! Descartes responded: Oh yeah? Try this: $x^3 + y^3 = 3xy$
Folium of Descartes

Use the method of implicit differentiation to find \( \frac{dy}{dx} \) for

\[ x^3 + y^3 = 3xy \]

(Please pause the video and try it yourself.)
Folium of Descartes

Use the method of implicit differentiation to find \( \frac{dy}{dx} \).

Step 1. Take derivatives on both sides.

\[
x^3 + y^3 = 3xy
\]

Chain rule:
\[
\frac{d}{dx}[y^3] = 3y^2 \frac{dy}{dx}
\]

Product and Chain rules:
\[
\frac{d}{dx}[xy] = (1)y + x \frac{dy}{dx}
\]

\[
3x^2 + 3y^2 \frac{dy}{dx} = 3(1)y + 3x \frac{dy}{dx}
\]
Folium of Descartes

Step 2. Solve for \(\frac{dy}{dx}\):

\[
3x^2 + 3y^2 \frac{dy}{dx} = 3(1)y + 3x \frac{dy}{dx}
\]

\[
x^2 + y^2 \frac{dy}{dx} = y + x \frac{dy}{dx}
\]

\[
y^2 \frac{dy}{dx} - x \frac{dy}{dx} = y - x^2 \quad \text{so} \quad \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}
\]
Find the mistake:

Use implicit differentiation on

\[ 2y + \cos x + x^2y + \sin(4y) = 1 \]

Step 1: We take the derivative with respect to \( x \):

INCORRECT answer:

\[ 2 - \sin x + 2x \frac{dy}{dx} + 4 \cos(4y) \frac{dy}{dx} = 1 \]

Where are the three mistakes?
(Please pause the video and look for them.)
Use implicit differentiation on

\[ 2y + \cos x + x^2y + \sin(4y) = 1 \]

Step 1: We take the derivative with respect to \( x \):

**INCORRECT** answer:

\[ 2 - \sin x + 2x \frac{dy}{dx} + 4 \cos(4y) \frac{dy}{dx} = 1 \]

- forgot to use chain rule
- incorrect use of product rule
- derivative of a constant is 0
Use implicit differentiation on

\[ 2y + \cos x + x^2y + \sin(4y) = 1 \]

Step 1: We take the derivative with respect to x:

INCORRECT answer:

\[ 2 - \sin x + 2x \frac{dy}{dx} + 4 \cos(4y) \frac{dy}{dx} = 1 \]

CORRECT answer:

\[ 2 \frac{dy}{dx} - \sin x + 2xy + x^2 \frac{dy}{dx} + 4 \cos(4y) \frac{dy}{dx} = 0 \]