Math 221
Week 4 part 3

The Chain Rule
and a step by step
approach to word problems
Please take a moment to just breathe.
In this section:

We discuss the chain rule.

We give a general strategy for word problems.
The Chain Rule for the taking derivative of a composite function:

\[ [f(g(x))]' = f'(g(x)) \cdot g'(x) \]

- **f** is the “outside” function
- **g** is the “inside” function
- Evaluate **f’** at **g(x)**
- Evaluate **g’** at **x**
Alternative way to write it:

$$\frac{d(fg)}{dx} \bigg|_{x=a} = \frac{df}{dg} \bigg|_{g=g(a)} \frac{dg}{dx} \bigg|_{x=a}$$

Evaluate $f'$ at $g(a)$

Evaluate $g'$ at $a$
Example: compute \( \frac{d}{dx} \sin(x^2) \).

The outside function is \( f(x) = \sin x \)
The inside function is \( g(x) = x^2 \)

\[
\frac{d}{dx} \sin(x^2) = [f(g(x))]' = f'(g(x)) \cdot g'(x) = \cos(x^2)(2x)
\]
It’s important to be careful with your parentheses and inputs.

Compute \( \frac{d}{dx} \sin(x^2 + x) \).

Correct answer: \( \cos(x^2 + x)(2x + 1) \)

What is **wrong** with these two answers?

**Wrong answer number 1:** \( \cos(x^2 + x)2x + 1 \)

**Wrong answer number 2:** \( \cos(x)(2x + 1) \)
Compute \( \frac{d}{dx} \sin(x^2 + x) \).

Wrong answer number 1: \( \cos(x^2 + x)2x + 1 \)

There should be parentheses around the quantity \((2x + 1)\). Instead, we’ve written “\( \cos(x^2 + x)2x \) plus 1”

Wrong answer number 2: \( \cos(x)(2x + 1) \)

Here, \( \cos x \) should have been \( \cos(x^2 + x) \).

Correct answer: \( \cos(x^2 + x)(2x + 1) \)
Intuition behind the Chain Rule: \[ (f(g(x)))' = f'(g(x)) \cdot g'(x) \]

Suppose \( x = a \).

If \( x \) changes by \( \Delta x \),
then \( \Delta g \approx g'(a) \Delta x \)
and \( \Delta f \approx f'(g(a)) \Delta g \) by our Very Useful Fact.

So \( \frac{\Delta f}{\Delta x} = \frac{\Delta f}{\Delta g} \cdot \frac{\Delta g}{\Delta x} \approx f'(g(a))g'(a) \).

It remains to check that the error terms behave nicely. (We’ll leave that to the mathematicians.)
Example.
A 10 ft radius circular pond is slowly freezing from the outside in.

Starting at midnight, the radius of the open water decreases by 4 inches (1/3 foot) every hour.

At 3am, how fast is the open area shrinking?

Try setting this up on your own. (Pause the video!)
A 10 ft radius circular pond is slowly freezing from the outside in.

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At 3am, how fast is the open area shrinking?

**Step 1: Give things names**

$r(t)$ is the radius of open water at time $t$ after midnight

$A(r)$ is the area when the radius is $r$
Step 2: write the problem down mathematically

A 10 ft radius circular pond is slowly freezing from the outside in. Starting at midnight, the radius of the open water decreases by 4 inches (1/3 foot) every hour.

\[ r(t) = 10 - t/3 \quad \text{and} \quad dr/dt = - \frac{1}{3} \]

At 3am, how fast is the open area shrinking?

Find \( \frac{d}{dt} A(r(t)) \) at \( t = 3 \).

We’ll also need some formulas relating area and radius:

\[ A = \pi r^2 \quad dA/dr = 2\pi r \]
Step 3. Find the derivative:

\[
\frac{d}{dt} A(r(t)) = \frac{dA}{dr} \frac{dr}{dt} = 2\pi r(t) \ r'(t) \quad \text{using the chain rule}
\]

\[
= 2\pi(10 - t/3) \ (-1/3)
\]

Step 4. Plug in given values.

at \( t = 3, \quad \frac{d}{dt} A(r(t)) = -2\pi(10 - 3/3)/3 = -6\pi. \)
Example.
A rocket is launched vertically and observed from a radar 5 miles away. If the angle between horizontal and the line of sight to the rocket is increasing by 3 degrees a second when the angle is 60 degrees, how fast is the rocket rising at that time?

Try setting this up on your own. (Pause the video!)
Example.
A rocket is launched vertically and observed from 5 miles away. If the angle between horizontal and the line of sight to the rocket is increasing by 3 degrees a second when the angle is 60 degrees, how fast is the rocket rising at that time?

**Step 1. Give things names**

Let $\theta$ be the angle from horizontal to the line of sight. It is important to measure $\theta$ in radians.

Let $y$ be the height of the rocket, measured in miles.

Then $\tan \theta = y/5$. 
Example.
A rocket is launched vertically and observed from 5 miles away. If the angle between horizontal and the line of sight to the rocket is increasing by 3 degrees a second when the angle is 60 degrees, how fast is the rocket rising at that time?

**Step 2. Write the problem down mathematically.**

\[ y = 5 \tan \theta \]

When \( \theta = \pi/3 \), \( \frac{d\theta}{dt} = \pi/60 \) radians per second.

We want to know \( \frac{dy}{dt} \) when \( \theta = \pi/3 \).

The units of \( \frac{dy}{dt} \) will be miles per second.
Step 3. Find the derivative.

\[ y = 5 \tan \theta \]

\[
\frac{dy}{dt} = \frac{dy}{d\theta} \frac{d\theta}{dt} = (5 \sec^2 \theta) \frac{d\theta}{dt}
\]

Step 4. Plug in the numbers.

When \( \theta = \pi/3 \), we know \( \frac{d\theta}{dt} = \pi/60 \) degrees per second.

\[
\frac{dy}{dt} = 5 \sec^2(\pi/3)(\pi/60) = 5(2)^2(\pi/60) = \pi/3
\]

which is a bit more than 1 mile per second.