In this section:

We review the trigonometric functions.

We prove that \( \lim_{h \to 0} \frac{\sin h}{h} = 1. \)

We derive the derivative of several trigonometric functions.
We will show that
\[ \frac{d}{dt} \sin t = \cos t. \]

position of the mass:
\[ p(t) = \sin t \]

velocity of the mass:
\[ p'(t) = \cos t \]

To compute the derivatives, we will need the following limits.

\[ \lim_{h \to 0} \frac{\sin h}{h} = 1 \quad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0 \]

As evidence of the first limit, recall that the arc of a unit circle of angle \( \theta \) has length \( \theta \). The line segment representing \( \sin \theta \) appears to have nearly the same length as the bit of arc.

Proof using Squeeze Theorem that \( \lim_{h \to 0^+} \frac{\sin h}{h} = 1. \)

For \( h \) in the interval \([0, \pi/2]\),

\[ \sin h \leq h \leq \tan h, \text{ so } \]
\[ \frac{1}{\sin h} \leq \frac{h}{\sin h} \leq \frac{1}{\cos h} \]

\[ 1 \leq \frac{h}{\sin h} \leq \frac{1}{\cos h} \]

By the Squeeze Theorem, since \( \lim_{h \to 0^+} \frac{1}{\cos h} = 1, \)

\[ \lim_{h \to 0^+} \frac{h}{\sin h} = 1. \]

The proofs that

\[ \lim_{h \to 0^-} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0 \]

are similar.

We will also need these formulas:

\[ \sin(x + h) = \sin x \cos h + \cos x \sin h \]
\[ \cos(x + h) = \sin x \sin h - \cos x \cos h \]
Use the limit definition to compute the derivative of \( \sin x \):

\[
\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}
\]

\[
= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}
\]

\[
= \lim_{h \to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h}
\]

\[
= \lim_{h \to 0} \left[ \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right] = \cos x
\]

We have

\[
\frac{d}{dx} \sin x = \cos x
\]

and a similar computation shows

\[
\frac{d}{dx} \cos x = -\sin x
\]

Exercise: compute \( \frac{d}{dx} \tan x \) using the quotient rule.

(Please pause the video and try it yourself!)

\[
\frac{d}{dx} \tan x = \frac{d}{dx} \left[ \frac{\sin x}{\cos x} \right]
\]

\[
= \frac{(\sin x)'(\cos x) - (\sin x)(\cos x)'}{\cos^2 x}
\]

\[
= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}
\]

\[
= \frac{(\cos x)^2 + (\sin x)^2}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\]

**Trigonometric functions and their derivatives.**

\[
\begin{align*}
\sin x & \quad \cos x \\
\tan x & \quad \sec^2 x \\
\sec x & \quad \sec x \tan x \\
\cos x & \quad -\sin x \\
\cot x & \quad -\csc^2 x \\
\csc x & \quad -\csc x \cot x
\end{align*}
\]

If a trigonometric function starts with “co”, then its derivative has a negative sign.

Please memorize these.
A side note about \( \lim_{h \to 0} \frac{\sin h}{h} = 1 \)

In practice, the approximation \( \sin h \approx h \) for small \( h \) is very useful.

If we want to speed up the oscillation, we also have

\[
\lim_{h \to 0} \frac{\sin(2h)}{2h} = 1, \quad \text{etc.}
\]

Here is the full “Taylor series” for \( \sin x \) from Calc II.

\[
\sin h = h - h^3/3! + h^5/5! - h^7/7! - h^9/9! + \ldots
\]

The more terms you take, the better the approximation will be.