In this section:

We introduce the derivative function $f'(x)$.

We discuss what we can learn about $f(x)$ from the graph of $f'(x)$.

We define differentiability.

Definition. The **derivative function** $f'(x)$ of $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

for all $x$ where the limit exists.

For example, suppose an object is moving in a straight line. If $p(t)$ is its location at time $t$, then $p'(t)$ is its velocity at time $t$. Please take a moment to just breathe.
Example:

Compare \( f(x) = x^3 - x \) and its derivative \( f'(x) = 3x^2 - 1 \).

On the intervals \((-\infty, -1/\sqrt{3})\) and \((1/\sqrt{3}, \infty)\), the derivative \( f'(x) > 0 \), so the slope of the tangent line is positive, and \( f \) is increasing.

On the interval \((-1/\sqrt{3}, 1/\sqrt{3})\), the derivative \( f'(x) < 0 \), so the slope of the tangent line is negative, and \( f \) is decreasing.
Some other rates of change:

<table>
<thead>
<tr>
<th>Rate</th>
<th>Increasing</th>
<th>Decreasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>position</td>
<td>forward</td>
<td>backwards</td>
</tr>
<tr>
<td>population</td>
<td>growing</td>
<td>shrinking</td>
</tr>
<tr>
<td>stock price</td>
<td>gaining $$</td>
<td>losing $$</td>
</tr>
<tr>
<td>height</td>
<td>getting taller</td>
<td>getting shorter</td>
</tr>
<tr>
<td>temperature</td>
<td>getting warmer</td>
<td>getting cooler</td>
</tr>
</tbody>
</table>

Definition

We say that \( f(x) \) is **differentiable** at \( x = a \) if the limit

\[
\lim_{{h \to 0}} \frac{f(a + h) - f(a)}{h}
\]

exists.

For most functions that we see in this class, \( f(x) \) will be differentiable on its entire domain.

However, if \( f \) is discontinuous at \( x = a \), or if \( f \) has a sharp corner at \( x = a \), then \( f \) will not be differentiable at \( x = a \).

Example. Here are three ways to write the absolute value function:

\( f(x) = |x| \)

\( f(x) = \sqrt{x^2} \)

\( f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \)

Notice that this function is **continuous** everywhere since \( \lim_{{x \to 0}} f(x) \) exists and equals \( f(0) \).

Is it differentiable at \( x = 0 \)?

At \( x = 0 \), we consider the one-sided derivative limits as \( h \) approaches zero from the right and left:

If \( h \) approaches zero from above, then \( h > 0 \), and

\[
\lim_{{h \to 0^+}} \frac{|0 + h| - |0|}{h} = \lim_{{h \to 0^+}} \frac{h}{h} = 1
\]

If \( h \) approaches zero from below, then \( h < 0 \), and

\[
\lim_{{h \to 0^-}} \frac{|0 + h| - |0|}{h} = \lim_{{h \to 0^-}} \frac{-h}{h} = -1
\]

Since the one sided limits don’t agree, the two-sided limit does not exist.
Here is $|x|$ again, and its derivative.

$$\begin{align*}
f(x) &= \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0
\end{cases} \\
\end{align*}$$

$$\begin{align*}
f'(x) &= \begin{cases} 
  1 & \text{if } x > 0 \\
  -1 & \text{if } x < 0 \\
  \text{undefined} & \text{if } x = 0
\end{cases}
\end{align*}$$

The function $|x|$ is not differentiable at $x = 0$.

Theorem:

If a function is differentiable at $x = a$, then it is continuous at $x = a$.

The converse is not true, as we just saw.

A short digression into LOGIC:

Definition: The converse of the sentence “If A then B” is the sentence “If B then A”.

For example, the converse to “If you eat chili peppers, your mouth will hurt.”

is “If your mouth hurts, you are eating chili peppers.”

Logically, the converse sentence does not follow from the original sentence.

You can believe that eating chili peppers makes your mouth hurt AND believe that other things can make your mouth hurt, like sipping really hot tea.

The contrapositive of the sentence “If A then B” is the sentence “If not B, then not A”.

For example, the contrapositive to “If you eat chili peppers, your mouth will hurt.”

is “If you mouth does not hurt, you are not eating chili peppers.”

The contrapositive of a sentence is logically equivalent to the original. If the original is true, then the contrapositive must also be true.

If you believe that eating chili peppers always makes your mouth hurt, and your mouth doesn’t hurt, you can safely conclude that you are not eating chili peppers.
### Notation

Here are some ways to write the derivative of $f(x)$:

- $f'(x)$
- $\frac{df}{dx}$
- $\frac{d}{dx} f$
- $f$ (in physics)
- $\frac{dy}{dx}$ if $y = f(x)$

### Higher derivatives

Given a function $f$ we define the

- first derivative: $f' = \frac{df}{dx}$
- second derivative $f'' = \frac{d}{dx} \left[ \frac{df}{dx} \right] = \frac{d^2f}{dx^2}$
- third derivative $f''' = \frac{d}{dx} \left[ \frac{d^2f}{dx^2} \right] = \frac{d^3f}{dx^3}$
- fourth derivative $f^{(4)} = \frac{d}{dx} \left[ \frac{d^3f}{dx^3} \right] = \frac{d^4f}{dx^4}$ and so on.

### Very Useful Fact:

If $f$ is differentiable at $x = a$, then

$$f(a + h) = f(a) + f'(a)h + E(h)h,$$

where $E(h)$ is a function with limit 0 at 0.