Math 221
Week 3 part 2

Derivatives as slopes of tangent lines

In this section, we interpret the derivative as the slope of the tangent line, and learn how to find tangent lines.

Please take a moment to just breathe.

screenshot from https://www.youtube.com/watch?v=ht3DMx1spo
Review of two ways to write the equation of a line:

1. **Point-slope form**

\[ y - y_0 = m(x - x_0) \]

Here, \( m \) represents the slope, and \((x_0, y_0)\) is any point on the line.

2. **Slope-intercept form**

\[ y = mx + b \]

Here, \( m \) represents the slope, and the line goes through the point \((0, b)\).

Recall:

Definition. The **derivative** of the function \( p(t) \) at \( t = a \) is

\[ p'(a) = \lim_{h \to 0} \frac{p(a + h) - p(a)}{h} \quad \text{if this limit exists} \]

The tangent line to the graph of \( y = p(t) \) at \( t = a \) has slope \( p'(a) \).

Example: Find the tangent line to \( f(x) = x^2 \) at \( x = 1 \).

Step 1. Find the slope \( m = f'(1) \).
Step 2. Use point slope form with \((x_0, y_0) = (1, 1)\) and \( m \).

\[ f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{(1 + h)^2 - 1^2}{h} \]

\[ = \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} \]

\[ = \lim_{h \to 0} \frac{2h + h^2}{h} \]

\[ = \lim_{h \to 0} [2 + h] = 2 \]

So the slope of the line is \( m = 2 \).
Find the tangent line to \( f(x) = x^2 \) at \( x = 1 \).

The slope is \( m = f'(1) = 2 \), and the line goes through \((1,1)\).

\[
y - 1 = 2(x - 1)
\]

Example. Find the tangent line to \( f(x) = \frac{1}{x} \) at \( x = 2 \).

Try finding this yourself. (Please pause the video!)

Step 1: Find \( m = f'(2) \)

Step 2: Use \( m \) and \((x_0, y_0) = (2, \frac{1}{2})\) to write the line.

Example. Find the points where the tangent line to \( f(x) = x^3 - x \) are horizontal.

Try this yourself. (Please pause the video!)

\[
f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}
\]

\[
= \lim_{h \to 0} \frac{\frac{1}{2 + h} - \frac{1}{2}}{h} = \lim_{h \to 0} \frac{2}{2(2 + h)} \quad \frac{2}{2(2 + h)} = \lim_{h \to 0} \frac{-h}{2(2 + h)}
\]

\[
= \lim_{h \to 0} \frac{-1}{2(2 + h)} = -\frac{1}{4}
\]

Step 2: \((x_0, y_0) = (2, \frac{1}{2})\), so the line is

\[
y - \frac{1}{2} = -\frac{1}{4}(x - 2)
\]
Example. Find the points where the tangent line to \( f(x) = x^3 - x \) are horizontal.

1. Find the derivative at a general point \( x = a \)
2. Find values of \( a \) where the tangent line has slope 0 by solving \( f'(a) = 0 \).

1. Find the derivative at a general point \( x = a \)

\[
f'(a) = \lim_{h \to 0} \frac{[(a + h)^3 - (a + h)] - [a^3 - a]}{h}
\]

\[
= \lim_{h \to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a - h - a^3 + a}{h}
\]

\[
= \lim_{h \to 0} \frac{3a^2h + 3ah^2 + h^3 - h}{h}
\]

\[
= \lim_{h \to 0} [3a^2 + 3ah + h^2 - 1]
\]

\[
= 3a^2 - 1
\]

Step 2. A horizontal tangent line will have slope 0. Find the values of \( x \) where the tangent line has slope 0.

\[
f'(a) = 3a^2 - 1
\]

\[
f'(a) = 0 \text{ when } 3a^2 - 1 = 0
\]

Solve:

\[
3a^2 = 1
\]

\[
a = \frac{1}{\sqrt{3}} \approx 0.577
\]

\[
a = -\frac{1}{\sqrt{3}} \approx -0.577
\]

You may sometimes see this equivalent formulation for the derivative:

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \text{ if this limit exists.}
\]