Math 221
Week 3 part 1

Derivatives
as rates of change
Please take a moment to just breathe.
In this lecture, we introduce the derivative and use it to compute how fast a pumpkin thrown off the Altgeld bell tower will be traveling when it hits the ground.

Note to students:
Do NOT throw pumpkins (or anything else) off the Altgeld bell tower.
Suppose you drive up to Chicago. The graph below shows you miles north of Altgeld at time $t$.

- You forgot something and went back home.
- You hit traffic in Chicago.
Suppose your position at time $t$ is given by the function $p(t)$.

Then the average velocity on the time interval $(a, a + \Delta t)$ is

$$\frac{p(a + \Delta t) - p(a)}{\Delta t}.$$
Example: Average velocity

Suppose that on the interstate, driving through the cornfields, you drive at the local speed limit.

If you measure your position after 1 hour and after 3 hours, you get

\[ p(1) = 10 \]
\[ p(3) = 140 \]

Your average speed on the interstate is

\[ \frac{p(3) - p(0)}{3 - 1} = \frac{130}{2} \text{ mph, or 65 mph (about 105 kph)} \]
However, suppose that when you hit Chicago traffic, you slow down:

\[ p(3) = 140 \]
\[ p(4) = 160 \]

Your average speed then would be

\[
\frac{p(4) - p(3)}{4 - 3} = \frac{20}{1} = 20 \text{ mph} \ (\text{about 32 kph})
\]
How fast are you going right at $t = 3$? What does your speedometer say at 3pm?

You could compute your average speed in the one second between 3:00:00 and 3:00:01.

Or you could compute it over one millisecond, or one microsecond, or one nanosecond, or one picosecond....
The **instantaneous velocity** of an object with position $p(t)$, at a specific time $t = a$, is

$$
\lim_{\Delta t \to 0} \frac{p(a + \Delta t) - p(a)}{\Delta t}
$$

Note that this quantity also represents the slope of the tangent line to the graph of $y = p(x)$ at $(a, p(a))$. 
Definition. The derivative of the function $p(t)$ at $t = a$ is

$$p'(a) = \lim_{\Delta t \to 0} \frac{p(a + \Delta t) - p(a)}{\Delta t}$$ if this limit exists

If $p(t)$ represents the position of an object at time $t$, then $p'(a)$ represents the instantaneous velocity at time $t = a$.

The tangent line to the graph of $y = p(t)$ at $t = a$ has slope $p'(a)$. 
Example: Suppose we drop a pumpkin off of the Altgeld bell tower, and suppose its position at time $t$ is given by

$$p(t) = 40 - 4.9t^2$$

where $t$ is in seconds and $p(t)$ is in meters.

What is the pumpkin’s velocity at time $t = 1$?
If \( p(t) = 40 - 4.9t^2 \), find \( p'(1) \). We’ll use “\( h \)” for “\( \Delta t \).

\[
p'(1) = \lim_{h \to 0} \frac{p(1 + h) - p(1)}{h}
\]

\[
= \lim_{h \to 0} \frac{40 + 4.9(1 + h)^2 - 40 - 4.9(1)^2}{h}
\]

\[
= \lim_{h \to 0} \frac{4.9(1 + 2h + h^2) - 4.9}{h}
\]

\[
= \lim_{h \to 0} \frac{4.9(2h + h^2)}{h} = \lim_{h \to 0} 4.9(2 + h) = 9.8 \text{ m/s}
\]
To find $p'(a)$ for a more general time $t = a$:

$$
p'(a) = \lim_{h \to 0} \frac{p(a + h) - p(a)}{h}
$$

$$
= \lim_{h \to 0} \frac{40 + 4.9(a + h)^2 - 40 - 4.9a^2}{h}
$$

$$
= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2) - 4.9a^2}{h}
$$

$$
= \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h}
= \lim_{h \to 0} 4.9(2a + h) = 9.8a
$$
How fast is the pumpkin moving when it hits the ground?

\[ p(t) = 40 - 4.9t^2 \]

It hits the ground when

\[ 40 - 4.9t^2 = 0, \]

which happens at \( t = \sqrt{\frac{40}{4.9}} \).

The velocity at that time is

\[ p'(\sqrt{\frac{40}{4.9}}) = 9.8 \sqrt{\frac{40}{4.9}} = 28 \text{ m/s} \]

or about 101 kph or 63 mph.