Math 221
Week 2 part 4

Continuity
Please take a moment to just breathe.
In English, the word “continuous” means “without interruption.”

In this section,

we give the mathematical definition of continuity

we give some uses of continuity
Definition:
We say the function $f(x)$ is continuous at $x = a$ if

1. $f(x)$ is defined at $x = a$

2. $\lim_{{x \to a}} f(x)$ exists, and

3. $\lim_{{x \to a}} f(x) = f(a)$
Example: GAP

1. $f(x)$ is defined at $x = a$

2. $\lim_{{x\to a}} f(x)$ exists

3. $\lim_{{x\to a}} f(x) = f(a)$
Example: Jump

1. $f(x)$ is defined at $x = a$

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$
Example: vertical asymptote

1. $f(x)$ is defined at $x = a$

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

All false!
A function is said to be continuous on the open interval \((b, c)\) if it is continuous at every point in the interval.

A function is said to be continuous on the closed interval \([b, c]\) if it is continuous at every point in \((b, c)\), and
\[
\lim_{x \to b^+} f(x) = f(b), \quad \text{and} \quad \lim_{x \to c^-} f(x) = f(c).
\]

Most functions we will see in this class are continuous on their domains.
Example: Is this function continuous everywhere?

\[ f(x) = |x| = \begin{cases} 
-x & \text{for } x < 0 \\
0 & \text{for } x = 0 \\
x & \text{for } x > 0 
\end{cases} \]

1. \(|0|\) is defined.
2. \(\lim_{x \to 0} |x|\) exists (since one sided limits agree)
3. \(\lim_{x \to 0} |x| = |0|\)

Yes!
Example: removable discontinuity

\[ f(x) = \frac{x^2 - 4}{x - 2} \] is not continuous at \( x = 2 \)

since \( f(2) \) is not defined.

However, since \( \lim_{x \to 2} f(x) = 4 \), we can "fix" the discontinuity by adding in a value for \( f(2) \):

\[ f(x) = \begin{cases} 
\frac{x^2 - 4}{x - 2} & \text{for } x \neq 2 \\
4 & \text{for } x = 2 
\end{cases} \]
Example: Is this function continuous? If not, which of the three conditions does it fail to satisfy?

\[ f(x) = \begin{cases} 
  \frac{x^2 - 4}{x - 2} & \text{for } x \neq 2 \\
  10 & \text{for } x = 2 
\end{cases} \]

Please pause the video and try this yourself.
This function is not continuous. It satisfies the first two conditions, but not the third.

\[ f(x) = \begin{cases} 
\frac{x^2 - 4}{x - 2} & \text{for } x \neq 2 \\
10 & \text{for } x = 2 
\end{cases} \]

1. \( f(x) \) is defined at \( x = a \)

2. \( \lim_{x \to a} f(x) \) exists

3. \( \lim_{x \to a} f(x) = f(a) \) \hspace{1cm} \text{False!}
Example: **removable discontinuity**

\[ f(x) = x^2 \sin(1/x) \] is not continuous at \( x = 0 \) since \( f(0) \) is not defined.

However, since \( \lim_{x \to 0} f(x) = 0 \), we can "fix" the discontinuity by adding in a value for \( f(0) \):

\[ f(x) = \begin{cases} 
  x^2 \sin(1/x) & \text{for } x \neq 0 \\
  0 & \text{for } x = 0 
\end{cases} \]
Example: \( f(x) = \frac{|x|}{x} \) is not continuous at \( x = 0 \).

\[
\lim_{x \to 0^+} \frac{|x|}{x} = 1
\]

\[
\lim_{x \to 0^-} \frac{|x|}{x} = -1.
\]

Since the left and right sided limits at \( x = 0 \) do not agree, there is no way to fix this discontinuity.
Example. \( f(x) = \frac{1}{x^2} \) is not continuous at \( x = 0 \).

Since \( \lim_{x \to 0} \frac{1}{x^2} = \infty \), there is no way to fix this discontinuity.
Some uses of continuity:

1. If $f$ is continuous, then we can move limits “inside” of $f$:

If $\lim_{x \to a} g(x) = L$ and $f(x)$ is continuous at $L$,

then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(L)$
Example:

\[
\lim_{x \to 0} [e^{\arctan(x)}] = e^{\lim_{x \to 0} \arctan(x)}
\]

(since \( e^x \) is continuous everywhere)

\[
= e^{\arctan(\lim_{x \to 0} [x])}
\]

(since \( \arctan x \) is continuous at 0)

\[
= e^{\arctan(0)} = e^0 = 1
\]
2. The Intermediate Value Theorem:

Suppose there is a flood and water is rising in a river.
At 1pm, the water level is 4ft.
At 3pm, the water level is 7ft.

Was the water level ever exactly at 6ft?

Yes, because the water level is a continuous function of time.
The Intermediate Value Theorem:

Suppose \( f \) is continuous on the closed interval \([a, b]\), and \( w \) is a value strictly between \( f(a) \) and \( f(b) \).

Then for some \( c \) in the open interval \((a, b)\), \( f(c) = w \).
Example.
Show $f(x) = x^3 - x - 2$ has a root in $(1,2)$.

$f(1) = -2$
$f(2) = 4$

Since 0 is between $-2$ and 4, there must be some $c$ in $(1,2)$ so that $f(c) = 0$, by the Intermediate Value Theorem.