Please take a moment to just breathe.
in this section:

We precisely define “limit 0 at 0” using infinitesimally small numbers $\epsilon$ and $\delta$.

We extend the definition to more general limits.

We also give some rules for combining functions with limit 0 at 0.

This approach is not in the textbook, but can be found on worksheet 1 of week 2.
Informal Definition of “limit zero at zero”

Let $E(h)$ be a function defined on an open interval about 0, except possibly at 0.

Informally, we say

$E(h)$ has "limit 0 at 0"

if we can force $E(h)$ to be as close to 0 as we like by choosing $h$ close enough to 0.
Precise Definition of “limit zero at zero”

Let $E(h)$ be a function defined on an open interval about 0, except possibly at 0.

We say $E(h)$ has "limit 0 at 0" if

for any “challenge” $\epsilon > 0$,

there exists a “response” $\delta > 0$

so that if $|h| < \delta$ but $h \neq 0$,

then $|E(h)| < \epsilon$. 
for any “challenge” $\epsilon > 0$,
there exists a “response” $\delta > 0$
so that if $|h| < \delta$ but $h \neq 0$,
then $|E(h)| < \epsilon$. 
To understand the logic, consider the following game between Eddie and Deisha.

Each round, Eddie challenges with an $\epsilon$, and Deisha responds with a $\delta$ that works for that $\epsilon$, if she can.

Deisha wins if she can convince Eddie she has a strategy that will always work. If Deisha wins, the function has limit 0 at 0.

Eddie wins if he ever "stumps" Deisha by finding an $\epsilon$ that she can’t respond to. If Eddie wins, the function does not have limit 0 at 0.
Example. Show $E(h) = 2h$ has limit 0 at zero 0.

Suppose Eddie suggests $\epsilon = 0.4$. Then Deisha should respond with $\delta = 0.2$ since

If $|h| < 0.2$
then $|E(h)| = |2h| < 0.4$

Suppose Eddie suggests $\epsilon = 0.006$. Then Deisha should respond with $\delta = 0.003$ since

If $|h| < 0.003$
then $|E(h)| = |2h| < 0.006$

Deisha’s strategy should be to always let $\delta = \epsilon / 2$. 
Now we have a proof that $E(h) = 2h$ has limit 0 at 0.

We need to show:
for any “challenge” $\epsilon > 0$,
there exists a “response” $\delta > 0$
so that if $|h| < \delta$ but $h \neq 0$,
then $|E(h)| < \epsilon$.

Given Eddie’s challenge $\epsilon > 0$,
Deisha responds with $\delta = \epsilon/2$, and then
If $|h| < \delta$ but $h \neq 0$
then $|E(h)| = |2h| = 2|h| < 2\delta = 2(\epsilon/2) = \epsilon$
Example. Show $E(h) = h/3$ has limit 0 at zero 0.

1. What should Deisha’s strategy be?

2. Fill in the blanks in this proof.

Given $\epsilon > 0$,

Let $\delta = \ldots$. Then

If $|h| < \delta$ but $h \neq 0$

then $|E(h)| \ldots$

Please pause the video and try this yourself.
Example. Show $E(h) = h/3$ has limit 0 at zero 0.

1. What should Deisha’s strategy be?

$$\delta = 3\epsilon$$

2. Here is the proof:

Given $\epsilon > 0$,

Let $\delta = 3\epsilon$. Then

If $|h| < \delta$ but $h \neq 0$,

then $|E(h)| = |h/3| = |h|/3 < \delta/3 = \epsilon$
Definition of the more general limit.

If $f(x)$ is a function defined on an open interval about $a$, except possibly at $a$, and $L$ is a number, we say

$$\lim_{x \to a} f(x) = L$$

if

$$f(a + h) = L + E(h)$$

for some error function $E(h)$ that has limit 0 at 0.

So to check a limit, we need to check that

$$E(h) = f(a + h) - L$$

has limit 0 at 0.
Example. Show that \( \lim_{x \to 1}[2x + 3] = 5 \)

Here,
\[
f(x) = 2x + 3
\]
\[
a = 1
\]
\[
L = 5
\]

Step 1: Find \( E(h) = f(a + h) - L \)

\[
E(h) = f(1 + h) - 5 = 2(1 + h) + 3 - 5 = 2h
\]

Step 2: Show that \( E(h) = 2h \) has limit 0 at 0.

We did this earlier, so we’re done.
Here are some rules for limit 0 at 0:

Suppose $E(h)$ and $F(h)$ are defined for all $h$ on an open interval about 0, except possibly at 0. Then:

1. (Sum rule)

If $E$ and $F$ both have limit 0 at 0,
  then so does their sum $E + F$.

2. (Product rule)

If $E$ and $F$ both have limit 0 at 0,
  then so does their product $EF$. 
Some rules about limit 0 at 0

3. (Composition rule)

If $E(h)$ and $F(h)$ both have limit 0 at 0,
then so does their composition $E(F(h))$.

4. (Squeeze rule)

If $|E(h)| \leq |F(h)|$ for all $h$, and $F(h)$ has limit 0 at 0,
then $E(h)$ has limit 0 at 0.
Example: Prove that \( \lim_{x \to 3} x^2 + 1 = 10 \).

Let \( f(x) = x^2 + 1 \).

\[
E(h) = f(3 + h) - 10 \\
= (3 + h)^2 + 1 - 10 \\
= h^2 + 6h + 9 + 1 - 10 = h^2 + 6h
\]

It remains to show that \( E(h) = h^2 + 6h \) has limit 0 at 0.

1) We show \( 6h \) has limit 0 at 0.
2) We show \( h^2 \) has limit 0 at 0.
3) The sum rule says \( 6h + h^2 \) has limit 0 at 0.
5. (Bounded Product rule)

If $C$ is a positive constant, and $|F(h)| \leq C$ except possibly at 0, and $E(h)$ has limit 0 at 0, then the product $EF(h)$ has limit 0 at 0.

Example: Prove that $h^2 \sin(1/h)$ has limit 0 at 0.

Proof:

1) First show $h^2$ has limit 0 at 0.

2) Then since $|\sin(1/h)| \leq 1$ for all $h$ away from 0, $h^2 \sin(1/h)$ has limit 0 at 0 by the bounded product rule.