In the 1600s, Newton and Leibniz independently created new mathematical language to talk about tangent lines and rates of change.

We will focus on two questions to start:

**Question 1.** Given a graph of a function, how do we find its tangent lines?

**Question 2.** How do we describe the instantaneous velocity of an object in motion?
Question 1. Given a graph, how do we find its tangent lines?

Answer: Find secant lines that go through \((a, f(a))\) and \((a + h, f(a + h))\) for small values of \(h\). Then let \(h\) get “infinitesimally” small.

The slope of the secant line through \((a, f(a))\) and \((a + h, f(a + h))\) is

\[
\frac{f(a + h) - f(a)}{h}
\]

Newton and Liebnitz developed language to express what happens to

\[
\frac{f(a + h) - f(a)}{h}
\]

as \(h\) gets “infinitesimally” close to 0:

\[
\lim_{{h \to 0}} \frac{f(a + h) - f(a)}{h}
\]
Question 2. How do we describe the instantaneous velocity of an object in motion?

To find out how fast an object is moving:

Measure the distance it travels
Divide by the amount of time it takes:

If $\Delta x$ is the change in location, and $\Delta t$ is the change in time

then $\frac{\Delta x}{\Delta t}$ is the average velocity.

We then let $\Delta t$ get "infinitesimally" small, and see what happens to $\frac{\Delta x}{\Delta t}$.

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

Example.

Suppose you drop an apple off of the top of the Altgeld bell tower at time $t = 0$.

Ignoring wind resistance, the distance it has fallen by time $t$ is given by the formula

$$x = 4.9t^2$$

where $x$ is in meters and $t$ is in seconds.

How fast is the apple traveling at time $t = 1$?
We compute the average velocity on time interval from $t = 1$ to $t = 2$

\[ x(1) = 4.9(1) \]
\[ x(2) = 4.9(4) \]

\[ \frac{\Delta x}{\Delta t} = \frac{4.9(4) - 4.9(1)}{1} = 14.7 \]

We compute the average velocity on the smaller time interval from $t = 1$ to $t = 1.1$

\[ x(1) = 4.9(1) \]
\[ x(1.1) = 4.9(1.21) \]

\[ \frac{\Delta x}{\Delta t} = \frac{4.9(1.21) - 4.9(1)}{0.1} = 10.29 \]

Even smaller time interval: (1,1.01)

\[ x(1) = 4.9(1) \]
\[ x(1.01) = 4.9(1.0201) \]

\[ \frac{\Delta x}{\Delta t} = \frac{4.9(1.0201) - 4.9(1)}{0.01} = 9.849 \]

Data:

<table>
<thead>
<tr>
<th>time interval</th>
<th>average speed on that interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2)</td>
<td>14.7</td>
</tr>
<tr>
<td>(1,1.1)</td>
<td>10.29</td>
</tr>
<tr>
<td>(1,1.01)</td>
<td>9.849</td>
</tr>
<tr>
<td>(1,1.001)</td>
<td>9.8049</td>
</tr>
<tr>
<td>(1,1.0001)</td>
<td>9.80049</td>
</tr>
</tbody>
</table>

Guess: instantaneous velocity of the apple at $t = 1$ is 9.8 meters per second.

It would be very convenient if we had an algorithm to compute these velocities from just the formula.