Math 302

Regular Tilings

A $n$-gon is a polygon with $n$ sides. A polygon is regular if all its sides have the same length and all its internal angles have the same measure.

A regular \( \{n, k\} \) tiling of a space $M$ is made by taking identical copies of a regular $n$-gon and using these $n$-gons to cover every point in $M$ so that there are no overlaps except that each edge of one $n$-gon is also the edge of another $n$-gon and so that each vertex is the vertex of $k$ $n$-gons. The $n$-gons are called faces or tiles of the tiling.

The dual tiling to a given tiling $T$ is made by putting a vertex in the center of each face of $T$ and drawing an edge between two such vertices when they correspond to adjacent faces. Faces of the dual tiling will correspond to vertices of the original tiling.

Part 1. The Euclidean Plane

1. Which regular \( \{n, k\} \) tilings work on $E^2$? What are their duals?

Part 2. The sphere

1. Which regular \( \{n, k\} \) tilings work on $S^2$? What are their duals?

A polyhedron is a three-dimensional version of a polygon. It is a region of space with flat walls. In fact, each wall will be a polygon, so we can construct polyhedra by gluing polygons together along their edges, as we did with the plastic construction toys.

The polygons are called the faces of the polyhedron, the sides are called their edges, and their corners are called vertices.

2. How many regular polyhedrons can we build? What does this question have to do with the previous question?

Part 3. The hyperbolic plane

1. Which regular \( \{n, k\} \) tilings work on $H^2$? What are their duals? Look at Escher pictures or your math 302 syllabus for some examples.
Part 4. Angle Sum and Classifying Tilings according to Curvature

We would like to find a simple numerical criterion to tell if a regular $\{n, k\}$ tiling is hyperbolic, spherical, or Euclidean, without having to memorize the lists above. You may derive such a criterion yourself; or, if you’d like, you may follow the steps below.

1. What do we know about how large the angle sum of a triangle can be in $H^2$, $S^2$, and $E^2$? What about the total angle sum of an $n$-gon?

2. Let $\alpha_n$ be the interior angle of an regular $n$-gon. How big can $\alpha_n$ be? Do this for each of the three spaces.

3. In a regular $\{n, k\}$ tilings, $k$ tiles meet at a vertex, so the total angle sum around the vertex must be $2\pi$. Express this fact in terms of $k$ and $\alpha_n$.

4. Combine the information above to get a criterion to tell whether a regular $\{n, k\}$ tilings is hyperbolic, spherical, or Euclidean. Extra for finding the simplest possible expression!

Part 5. Higher Dimensions – Getting ready for the CAVE

So far, we have been looking at tiling two dimensional surfaces with regular polygons. You can also tile three dimensional surfaces with regular polyhedra.

1. How many ways can you tile three dimensional Euclidian space with regular polyhedra? For instance, how many cubes would have to fit together at a vertex? Could you use regular dodecahedrons?

2. How many ways can you tile the three dimensional sphere with regular polyhedra? What is the three dimensional sphere, anyway??

3. What is three dimensional hyperbolic space??