Math 302
Maps of the Hyperbolic Plane (Revised 3-7-01)

The Rectangular Coordinate Map

Let $H^2$ be the hyperbolic plane of radius $\rho$. This means that the radius of the inner circle of each annular piece in your model is $\rho$. Let $E^2$ be the standard Euclidean plane.

We define a map $\chi : E^2 \to H^2$, called the rectangular coordinate map, as follows.

First, pick any point $O \in H^2$. To find $\chi(u,v)$, start at $O$. Move distance $u$ to the right along the annular curve through $O$ (move left if $u < 0$). Next, move up a distance $v$ along a radial line (if $v < 0$ move down). Note that "up" is chosen so that the annular strips look like smiles, not frowns.

It takes vertical lines to radial lines, which are geodesics. Moreover, it preserves the length of these lines. Therefore, the vertical distortion is 1.

On the other hand, it takes horizontal lines to annular curves, which are not geodesics. Moreover, it does not preserve the length of these lines. It takes the horizontal line joining $(u,v)$ and $(u+du,v)$ to the annular curve that joins the radial lines through $u$ and $u+du$ at height $v$. This curve is bigger for positive $v$, and smaller for negative $v$. To be more precise, recall that $\rho$ is the inner radius of the annuli we used, and let $\delta$ be the width. The outer radius is then $\rho + \delta$. The length of the annular curve between two radial geodesics decays by a factor of $\rho / (\rho + \delta)$ each time we move up one strip. If we move up by distance $v$, we move $k = v/\delta$ strips, so the new length is $du$ times

$$\left( \frac{\rho}{\rho + \delta} \right)^k = \left( \frac{\rho}{\rho + \delta} \right)^{v/\delta} = \left( \frac{\rho + \delta}{\rho} \right)^{-v/\delta} = \left( 1 + \frac{\delta}{\rho} \right)^{-v/\delta}.$$

The true hyperbolic plane (as opposed to the approximate model) is obtained by letting $\delta \to 0$. Define

$$c = \lim_{\delta \to 0} \left( 1 + \frac{\delta}{\rho} \right)^{-v/\delta}.$$

Then

$$\ln c = \lim_{\delta \to 0} \frac{-v}{\delta} \ln \left( 1 + \frac{\delta}{\rho} \right).$$

Applying L'Hôpital's rule, $\ln c = \lim_{\delta \to 0} -v/(\rho + \delta) = -v/\rho$, so $c = e^{-v/\rho}$. This means that the path from $\chi(u,v)$ to $\chi(u + du, v)$ has length $e^{-v/\rho} du$. Therefore, the distortion for $\chi$ in the $u$-direction is $e^{-v/\rho}$. Thus, we see that the rectangular coordinate map is not conformal, because it distorts things differently in the $u$- and $v$-directions.

Recall that the hyperbolic plane can be locally (though not globally) embedded in three-space. Since all our computations are local, this means that we can do all our computations as if $\chi$ is a map to three-space, just like we did with the sphere.

In these terms, we now have

$$\left\| \frac{\partial \chi}{\partial u} \right\| = e^{-\frac{v}{\rho}} \quad \text{and} \quad \left\| \frac{\partial \chi}{\partial v} \right\| = 1.$$

Moreover, the map $\chi$ takes horizontal and vertical lines to paths that are perpendicular, so

$$\frac{\partial \chi}{\partial v} \perp \frac{\partial \chi}{\partial u}.$$

Finally, $\chi$ is one-to-one and onto because it has an inverse. Note that radial lines never intersect. Given a point $P$, let $v$ be the distance you have to travel down a radial line to hit the annular curve through $O$. Then let $u$ be the distance you have to travel left along this annular curve to hit $O$. 

The Upper-Half-Plane Map

Since the distortion of $\chi$ is different in the horizontal and vertical directions, it is not conformal. We will alter $\chi$ so that the horizontal and vertical distortions are equal, while at the same time keeping the property that it takes horizontal and vertical lines to perpendicular curves. Can we change the horizontal distortion to match the vertical distortion, $1$? No! If that were possible, then we would have an isometry from $\mathbb{E}^2$ to $\mathbb{H}^2$, and it can be shown that no such isometry exists. Instead, we will change the vertical distortion to match the horizontal distortion, $e^{-v/\rho}$. This is what we did to get the (conformal) Mercator projection from the (nonconformal) cylindrical projection.

Define a map $v : (-\infty, \infty) \times (0, \infty) \to (-\infty, \infty)$ by

$$v(x, y) = \rho \ln(y/\rho).$$

A direct computation shows that $v_y = \frac{\rho}{y}$. We also define $u : (-\infty, \infty) \times (0, \infty) \to (-\infty, \infty)$ by $u(x, y) = x$ (No change in the horizontal direction).

Let $U$ be the upper half-plane, that is

$$U := \{(x, y) \in \mathbb{E}^2 \mid y > 0\}.$$

We define the upper-half-plane map $z : U \to \mathbb{H}^2$ by

$$z(x, y) := \chi(u(x, y), v(x, y)).$$

We can now use the chain rule to compute the derivatives.

$$z_x = \chi_u u_x + \chi_v v_x = \chi_u$$

and

$$z_y = \chi_u u_y + \chi_v v_y = \frac{\rho}{y} \chi_v.$$

Therefore, we have

$$|z_x| = |\chi_u| = e^{-\frac{v}{\rho}} = e^{-\frac{\rho \ln(y/\rho)}{\rho}} = e^{-\ln(y/\rho)} = \left(\frac{y}{\rho}\right)^{-1} = \frac{\rho}{y}$$

and

$$|z_y| = \frac{\rho}{y} |\chi_v| = \frac{\rho}{y}.$$

Since $\chi_u \perp \chi_v$ and since $z_x$ and $z_y$ are scalar multiples of $\chi_u$ and $\chi_v$, we have $z_x \perp z_y$. Hence, by the Conformal Criterion Theorem, the upper-half-plane map is conformal. One may also show that the upper-half-plane map is also one-to-one and onto.