Math 302

Conformal Maps of $\mathbb{E}^2$: Definitions and Theorems

Let $\gamma: [a, b] \to \mathbb{E}^2$ be a path. We will assume that $\gamma$ is differentiable. Expressed in coordinates, $\gamma(t) = (x(t), y(t))$.

**Definition** (tangent vector). The tangent vector to $\gamma$ at a point $\gamma(t)$ is $\gamma'(t) = (x'(t), y'(t))$.

For example, if $\gamma(t) = (t, t^2)$, then the tangent vector at $\gamma(t)$ is $(1, 2t)$. At $t = 0$, for example, the tangent vector is $(1, 0)$.

**Definition** (angle between two paths). The angle between two intersecting paths is the angle between their tangent vectors at the point of intersection.

**Definition.** A map is **conformal at** a point $P$ in its domain if it preserves the measures of all angles at the point $P$. (note: this does not mean that the map must send $P$ to itself.)

It is possible for a map to be conformal at some points but not at all points.

**Definition** (conformal map). Let $F$ be a map from $\mathbb{E}^2$ to $\mathbb{E}^2$. $F$ is **conformal** if it is conformal at all points.

For example, $F(x, y) = (3y, 3x)$ is conformal (everywhere), but is not an isometry. $G(x, y) = (x^3, y^3)$ is not conformal, though it is conformal at some points, like $(1, 1)$.

**Definition** (distortion: see page 169 of text). Let $\gamma$ be a path, and let $F: \mathbb{E}^2 \to \mathbb{E}^2$ be a map, not necessarily conformal. Then the distortion of $F$ along $\gamma$ at the point $P = \gamma(t_0)$ is

$$\lim_{t \to t_0} \frac{\text{the length of the path } F(\gamma) \text{ from } F(\gamma(t_0)) \text{ to } F(\gamma(t))}{\text{the length of the path } \gamma \text{ from } \gamma(t_0) \text{ to } \gamma(t)}.$$ \[
\text{In coordinates, this is } \frac{\| \frac{\partial F}{\partial x} \cdot x'(t_0) + \frac{\partial F}{\partial y} \cdot y'(t_0) \|}{\sqrt{x'(t_0)^2 + y'(t_0)^2}}.
\]

This expression may also be written as a ratio of the lengths of tangent vectors:

$$\frac{\|(F\gamma)'(t_0)\|}{\|\gamma'(t_0)\|}.$$

**Theorem.** If a map is conformal at a point $P$, then at $P$, its distortion is the same in all directions.

**Theorem** (Conformal Criterion). Given a map $F$ and a point $P$ in the domain of $F$, consider two paths which intersect at right angles at $P$. Suppose that the images under $F$ of these paths intersect at right angles, and also that the distortions in the directions of these two paths are the same. Then $F$ is conformal at $P$.

**Theorem** (Corollary of Criterion). Let $F$ be a map from $\mathbb{E}^2$ to $\mathbb{E}^2$. If, at a point $P$, the vectors $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ have the same length and are perpendicular to each other, then $F$ is conformal at $P$.

Note that the length of $\frac{\partial F}{\partial x}$ is the distortion of $F$ in the horizontal direction and the length of $\frac{\partial F}{\partial y}$ is the distortion of $F$ in the vertical direction.
Math 302

Conformal Maps from One Surface to Another:
Definitions and Theorems

You will notice that the definitions and theorems on this page are very similar to those on the previous page (Conformal Maps of $\mathbb{E}^2$). Here, though, the setting is more general; the maps go from one surface to another, not necessarily $\mathbb{E}^2$. We do assume, when necessary, that the surfaces are embedded in 3-space $\mathbb{R}^3$.

Let $\gamma : [a, b] \to M$ be a path. We will assume that $\gamma$ is differentiable. Expressed in coordinates, $\gamma(t) = (x(t), y(t), z(t))$, where these are the usual coordinates on $\mathbb{R}^3 \supset M$.

**Definition** (tangent vector). The **tangent vector** to $\gamma$ at $\gamma(t)$ is $\gamma'(t) = (x'(t), y'(t), z'(t))$.

**Definition** (angle between two paths). The **angle** between two intersecting paths is the angle between their tangent vectors at the point of intersection.

**Definition.** A map is **conformal** at a point $P$ in its domain if it preserves the measures of all angles between paths at the point $P$. (Note: this does not mean that the map must send $P$ to itself; the map may be from one surface to another.)

**Definition** (conformal map). Let $F$ be a map from $M$ to $M'$. $F$ is **conformal** if it is conformal at all points.

**Definition** (Distortion — see page 169 of text). The **distortion** of $F$ at a point $P$ in a given direction is the amount of stretching in that direction, that is, the factor by which the length of a path through $P$ is stretched at that point.

Let $\gamma$ be a path, and let $F : M \to M'$ be a map, not necessarily conformal. Then the **distortion** of $F$ along $\gamma$ at the point $P = \gamma(t_0)$ is

$$\lim_{t \to t_0} \frac{\text{the length of the path } F(\gamma)\text{ from } F(\gamma(t_0))\text{ to } F(\gamma(t))}{\text{the length of the path } \gamma\text{ from } \gamma(t_0)\text{ to } \gamma(t)}.$$

We will now give formulas for distortion in the setting of a map $F : \mathbb{E}^2 \to M'$; we will use rectangular coordinates $(u, v)$ on $\mathbb{E}^2$. Then $\partial F/\partial u$ is the image under $F$ of the unit tangent vector in the horizontal direction and $\partial F/\partial v$ is the image under $F$ of the unit tangent vector in the vertical direction. Therefore, the distortion of $F$ in the horizontal direction is $\|\partial F/\partial u\|$ and the distortion of $F$ in the vertical direction is $\|\partial F/\partial v\|$.

**Theorem** (Conformal Criterion). Given a map $F : M \to M'$ and a point $P \in M$, consider two paths which intersect at $P$ at right angles. Suppose that the images under $F$ of the two paths intersect at right angles, and also that the distortions in the directions of these two paths are the same. Then $F$ is conformal at $P$.

**Theorem** (Corollary of the Criterion). Let $F$ be a map from $\mathbb{E}^2$ to $M' \subset \mathbb{R}^3$. Use coordinates $(u, v)$ on $\mathbb{E}^2$, and $(x, y, z)$ on $M' \subset \mathbb{R}^3$. If, at a point $P$, the vectors $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ (computed as for any map $\mathbb{E}^2 \to \mathbb{R}^3$) have the same length and are perpendicular to each other, then $F$ is conformal at $P$. 