HW 8 Solutions

4.3
12 $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = 5x \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 5x - y = 0 \right\}$

i.e $W = \text{Nul}\left([5 \ 1] \right)$. Hence its a subspace of $\mathbb{R}^2$.

To find ~~the~~ a basis let $y = k$ $\therefore x = k/5$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k/5 \\ k \end{pmatrix} = k \begin{pmatrix} 1/5 \\ 1 \end{pmatrix}$$

$$\therefore W = \text{span}\left\{ \begin{pmatrix} 1/5 \\ 1 \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 5 \end{pmatrix} \right\}.$$

15 For this problem form a matrix with ~~columns~~ the given vectors as its columns... then row reduce it to its echelon form and then pick the pivot columns from the original matrix. That would give us the basis.

20 Note that v_1 can be expressed as $3v_2 - 5v_3$, a linear combination of v_2 and v_3. Hence

$$H = \text{span}\{v_1, v_2, v_3\} = \text{span}\{v_2, v_3\}.$$

We now need to check if $\{v_2, v_3\}$ is a linearly independent set. It is! Hence a basis of H is $\{v_2, v_3\}$.

Note that v_2 can also be expressed as a linear combination of v_1, v_3 namely $v_2 = \frac{1}{3} v_1 + \frac{5}{3} v_3$

Hence $\{v_1, v_3\}$ can also form a basis of H.

Of course, you also need to check if $\{v_1, v_3\}$ is an independent set.

21 (a) True; since $c.\vec{v} = \vec{0} \Rightarrow c = 0$ (assuming $\vec{v} \neq 0$).

(b) False; $\{b_1, \dots, b_p\}$ might not be an independent set.

(c) True; A can be reduced to an identity matrix. where each column has a pivot.

(d) False; Its a spaning set which is as small as possible (small here refers to being independent)

(e) False; if A is row equivalent to B. then $A\vec{x} = 0$ and $B\vec{x} = \vec{0}$ have the same solutions. Hence dependencey / independency is preserved under row operations.

4.5

6 $W = \left\{ \begin{pmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$.

Note $W = \text{Col}\left(\begin{bmatrix} 3 & 6 & -1 \\ 6 & 2 & -2 \\ -9 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix} \right)$

So you need to find a basis for Col(A). where $A = \begin{bmatrix} 3 & 6 & -1 \\ 6 & 2 & -2 \\ -9 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix}$

For this we pick the pivot columns! (Do yourself).

7 $W = \{ (a,b,c) : a - 3b + c = 0,\ b - 2c = 0,\ 2b - c = 0 \}$

Note $W = \text{Nul}\left(\begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & -1 \end{bmatrix} \right)$ .... Now find basis as done in class.

9 $W = \left\{ \begin{pmatrix} a \\ b \\ a \end{pmatrix} : a, b \in \mathbb{R} \right\}$ Note the first and third entries are the same

Also note $\begin{pmatrix} a \\ b \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\therefore W = \text{col}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$ .... continue yourself..

19 (a) True; Since the pivot columns forms a basis of col A.

(b) False; The plane might not contain the origin.

(c) False; $\dim(\mathbb{P}_4) = 5$. An example of a basis is $\{1, t, t^2, t^3, t^4\}$.

(d) False; S might not span V. eg. $V = \mathbb{R}^3$, $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(e) True; Since $\dim(V) \le p$. If T were linearly independent then $\dim(V)$ would be $> p$; a contradiction.

21 Consider the equation

$$c_1(1) + c_2(1-t) + c_3(2-4t+t^2) + c_4(6-18t+9t^2-t^3) = 0.$$

Solve this: i.e

$$\begin{aligned} c_1 + c_2 + 2c_3 + 6c_4 &= 0 \\ -c_2 - 4c_3 - 18c_4 &= 0 \\ c_3 + 9c_4 &= 0 \\ -c_4 &= 0. \end{aligned}$$

Solve them to get $c_1 = c_2 = c_3 = c_4 = 0$

i.e $B = \{1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3\}$ forms a linearly independent set

Moreover $\dim(\mathbb{P}_3) = 4$, Hence B is a basis of $\mathbb{P}_3$.

23 Ignore this problem for now.

4.6 6 $A_{6\times 3}$ has rank 3

$\therefore$ $\dim(\text{Nul}(A)) = 3 - \text{rank}(A)$
$= 0$

$\dim(\text{Row } A) = \text{rank } A = 3$

$\text{rank}(A^T) = \dim \text{col}(A^T) = \dim \text{Row}(A) = 3$

7 Since $A_{4\times 7}$ has 4 pivot columns, $\dim(\text{col}(A)) = 4$

but ~~but i.e~~ $\text{col}(A)$ is a subspace of $\mathbb{R}^4$. Hence $\text{col}(A) = \mathbb{R}^4$.

Now $\text{rank}(A) + \dim \text{Nul}(A) = 7$

$\Rightarrow 4 + \dim(\text{Nul } A) = 7$

$\Rightarrow \dim(\text{Nul } A) = 3$

i.e Nul A is a 3-dimensional subspace of $\mathbb{R}^7$, but is not $\mathbb{R}^3$.

15 Since $\dim(\text{Nul } A) + \text{rank}(A) = 8$

~~Sma~~ Asking for the smallest possible dimension of Nul (A) is equivalent to asking the largest possible ~~dimension~~ rank A · or largest possible dim (Col A) or dim (Row A)

Note that Row(A) is a subspace of $\mathbb{R}^6$. Hence

$\text{rank}(A) \le 6$.

i.e $\dim(\text{Nul } A) \geq 8 - 6 = 2$.