HW 3 Solutions

1.5 (10) Let $A \sim \begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$. To describe all solutions of $A\vec{x} = \vec{0}$ in parametric form.

So $$A \sim \begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix} \overset{R_2' = R_2 - 2R_1}{\sim} \begin{bmatrix} 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e. $x_1 + 3x_2 + 0x_3 - 4x_4 = 0$.

So we take $x_2 = k$, $x_3 = l$, $x_4 = m$ as free variables.

So $x_1 = -3k + 4m$

$$\therefore \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3k+4m \\ k \\ l \\ m \end{pmatrix} = k\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + l\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + m\begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix};$$

$k, l, m \in \mathbb{R}$.

(14) $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3x_4 \\ 8 + x_4 \\ 2 - 5x_4 \\ x_4 \end{pmatrix}$. Let $x_4 = k$.

So $\vec{x} = \begin{pmatrix} 0 \\ 8 \\ 2 \\ 0 \end{pmatrix} + k\begin{pmatrix} 3 \\ 1 \\ -5 \\ 1 \end{pmatrix}$ which is a line in $\mathbb{R}^4$ passing through $\begin{pmatrix} 0 \\ 8 \\ 2 \\ 0 \end{pmatrix}$ and parallel to $\begin{pmatrix} 3 \\ 1 \\ -5 \\ 1 \end{pmatrix}$.

(23) (a) True; $\vec{x} = \vec{0}$ is always a solution for $A\vec{x} = \vec{0}$.

(b) False; It gives an implicit description of the solution set in parametric vector form.

(c) False; The homogeneous always has the trivial solution.

(d) False; It describes a line through $\vec{p}$ parallel to $\vec{v}$.

(e) False; The solution set could be empty

example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ i.e $\left\{ \begin{matrix} x_1 & & = 0 \\ & x_2 & = 0 \\ & 0 & = 1 \end{matrix} \right\}$

$A\vec{x} = \vec{b}$.

(24) (a) False; there is some entry which is non-zero.

(b) True; if $\vec{u}$ and $\vec{v}$ are not multiples of each other, then they give different directions. Hence their span, which is the set of all possible linear combinations gives a 2-dimensional plane. Moreover if we pick $x_2 = x_3 = 0$ we get $\vec{0}$ as a point on the plane.

(c) True; if $\vec{x} = \vec{0}$ is a solution then $A \cdot \vec{0} = \vec{0}$. but our system was $A\vec{x} = \vec{b}$ so $\vec{0} = \vec{b} \Rightarrow$ its a homogenous system

(d) True; by the parallelogram law of addition of vectors.

(e) False; only when $A\vec{x} = \vec{b}$ is consistent (i.e has a solution) to begin with.

1.7

6

$$A \to \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \underset{R_3' = R_1}{\overset{R_1' = R_3}{\sim}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ -4 & -3 & 0 \\ 5 & 4 & 6 \end{bmatrix} \underset{R_4' = R_4 - 5R_1}{\overset{R_3' = R_3 + 4R_1}{\sim}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{bmatrix}$$

$$\underset{R_4' = R_4 + 4R_2}{\overset{R_3' = R_3 - 3R_2}{\sim}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{-1} & 4 \\ 0 & 0 & \boxed{7} \\ 0 & 0 & 0 \end{bmatrix}$$

There are 3 pivots. Hence ~~it is~~ the columns of A are linearly independent.

Or in other words, solving for $A\vec{x} = \vec{0}$, gives as

$$\left.\begin{aligned} x_1 + 3x_3 &= 0 \\ -x_2 + 4x_3 &= 0 \\ 7x_3 &= 0 \end{aligned}\right\} \Rightarrow x_3 = 0, \quad x_2 = 0, \quad x_1 = 0.$$

Hence only the trivial solution.

Hence linear independence.

8 Linear dependence, since there are 4 vectors from $\mathbb{R}^3$ ($\because 4 > 3$)

17 Linear dependent; since $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is in the set

19 $v_1 = \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$; Note $v_1 + 4v_2 = 0$., hence linearly dependent.

(22) (a) True; If $\vec{u}$ and $\vec{v}$ are linearly dependent, it implies $c_1\vec{u} + c_2\vec{v} = \vec{0}$ for ~~$c_1 \neq 0$ & $c_2 \neq 0$~~ not both $c_1, c_2 = 0$.

i.e. $u = k\vec{v}$ i.e $\vec{u}$ is a ~~mult~~ scalar multiple of $\vec{v}$

Hence they lie on the same line through the origin.

Conversely if $\vec{u}$ and $\vec{v}$ lie on the same line thru the origin, it means $\vec{u} = k\vec{v}$ for some k.

i.e $1\cdot\vec{u} - k\cdot\vec{v} = \vec{0}$. Hence $\{\vec{u}, \vec{v}\}$ is a linearly dependent set of vectors.

(b) False; consider. $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$. Note there are 3 vectors in $\mathbb{R}^4$. but they are not linearly independent; since

$$1\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix} - 1\cdot\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

(c) True; if z is in Span$\{x, y\}$ $\Rightarrow$ $z = c_1\vec{x} + c_2\vec{y}$ for some c_1, c_2 in $\mathbb{R}$. i.e $c_1\vec{x} + c_2\vec{y} - 1\cdot\vec{z} = \vec{0}$.

Hence $\{x, y, z\}$ is a linearly dependent set.

(d) False: Consider. $\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\}$.